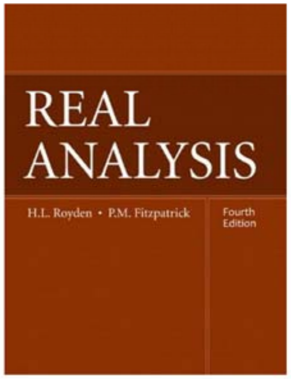
Real Analysis

Chapter 17. General Measure Spaces: Their Properties and Construction

17.5. The Carathéodory-Hahn Theorem—Proofs of Theorems



Proposition 17.11

Proposition 17.11. Let $\mathcal{S}$ be a collection of subsets of X and $\mu : \mathcal{S} \to [0, \infty]$ a set function. In order that the Carathéodory measure $\overline{\mu}$ induced by μ be an extension of μ (that is, $\overline{\mu} = \mu$ on $\mathcal{S}$) it is necessary that μ be both finitely additive and countably monotone and, if $\varnothing \in \mathcal{S}$, then $\mu(\varnothing) = 0$.

Proof. Let $(X, \mathcal{M}, \overline{\mu})$ be the Carathéodory measure space induced by μ and suppose $\overline{\mu} : \mathcal{M} \to [0, \infty]$ extends $\mu : \mathcal{S} \to [0, \infty]$. First, if $\varnothing \in \mathcal{S}$ then $\overline{\mu}(\varnothing) = 0$ since $\overline{\mu}$ is a measure (by the definition of measure, page 338) and $\mu(\varnothing) = \overline{\mu}(\varnothing) = 0$ since $\overline{\mu}$ extends μ. A measure is finitely additive by Proposition 17.6, so if $\{E_k\}_{k=1}^\infty \subset \mathcal{S}$ and $\cup_{k=1}^n E_k \in \mathcal{S}$, then $\overline{\mu}$ is finitely additive on the E_k's and so μ is finitely additive on the E_k's since $\overline{\mu}$ extends μ. For countable monotonicity, we must show that (by definition, see page 346) for each $E \in \mathcal{S}$ and each $\{E_k\}_{k=1}^\infty \subset \mathcal{S}$ with $E \subset \cup_{k=1}^\infty E_k$, $\mu(E) \le \sum_{k=1}^\infty \mu(E_k)$.

Proposition 17.11 (continued)

Proof (continued). For such E and $\{E_k\}_{k=1}^\infty$, we have $\mu^*(E) \le \sum_{k=1}^\infty \mu(E_k)$ by the definition of μ^*. For $E \in \mathcal{S}$ we have $\mu^*(E) = \inf \{\sum_{k=1}^\infty \mu(A_k)\} \le \mu(E)$ and

$$\begin{aligned} \mu(E) &= \overline{\mu}(E) \le \overline{\mu}\left(\cup_{k=1}^\infty A_k\right) \text{ since } \overline{\mu} \text{ is monotone} \\ &\le \sum_{k=1}^\infty \overline{\mu}(A_k) \text{ since } \overline{\mu} \text{ is countably monotone} \\ &= \sum_{k=1}^\infty \mu(A_k) \text{ since } \overline{\mu} \text{ extends } \mu. \end{aligned}$$

Taking an infimum over all such coverings of E we have $\mu(E) \le \mu^*(E)$, so that $\mu^*(E) = \mu(E)$. Hence $\mu(E) = \mu^*(E) \le \sum_{k=1}^\infty \mu(E_k)$ and μ is countably monotone. □

Theorem 17.12

Theorem 17.12. Let $\mu : \mathcal{S} \to [0, \infty]$ be a premeasure on a nonempty collection $\mathcal{S}$ of subsets of X that is closed with respect to the formation of relative complements. Then the Carathéodory measure $\overline{\mu} : \mathcal{M} \to [0, \infty]$ induced by μ is an extension of μ called the *Carathéodory extension* of μ.

Proof. Let $A \in \mathcal{S}$. We need to show A is measurable and $\mu(A) = \overline{\mu}(A)$. Let $\varepsilon > 0$. We show that for all $E \subset X$ with $\mu^*(E) < \infty$ that

$$\mu^*(E) + \varepsilon \ge \mu^*(E \cap A) + \mu^*(E \cap A^c) \qquad (8)$$

(the restriction $\mu^*(E) < \infty$ is justified by the finite monotonicity of μ^*; see page 347). By definition of outer measure in terms of infimum, there exists set $\{E_k\}_{k=1}^\infty$ of sets in $\mathcal{S}$ that covers E and such that

$$\mu^*(E) + \varepsilon \ge \sum_{k=1}^\infty \mu(E_k). \qquad (9)$$

Theorem 17.12 (continued 1)

Proof (continued). Since $\mathcal{S}$ is closed with respect to the formation of relative complements, then $E_k \cap A^c = E_k \setminus A \in \mathcal{S}$ and $E_k \cap A = E_k \setminus (E_k \setminus A) \in \mathcal{S}$ for all $k \in \mathbb{N}$. Since premeasures are finitely additive by definition, then $\mu(E_k) = \mu(E_k \cap A) + \mu(E_k \cap A^c)$ for all $k \in \mathbb{N}$, and so

$$\sum_{k=1}^{\infty} \mu(E_k) = \sum_{k=1}^{\infty} \mu(E_k \cap A) + \sum_{k=1}^{\infty} \mu(E_k \cap A^c). \tag{10}$$

Next, $\{E_k \cap A\}_{k=1}^{\infty}$ and $\{E_k \cap A^c\}_{k=1}^{\infty}$ are subsets of $\mathcal{S}$ which cover $E \cap A$ and $E \cap A^c$, respectively. So from the definition of outer measure and infimum,

$$\sum_{k=1}^{\infty} \mu(E_k \cap A) \geq \mu^*(E \cap A) \text{ and } \sum_{k=1}^{\infty} \mu(E_k \cap A^c) \geq \mu^*(E \cap A^c),$$

from (10),

Theorem 17.12 (continued 2)

Proof (continued).

$$\sum_{k=1}^{\infty} \mu(E_k) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c),$$

and from (9), this implies

$$\mu^*(E) + \varepsilon \geq \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

Since $\varepsilon > 0$ is arbitrary, we have (8) and so A is measurable.

Next, for any $A \in \mathcal{S}$ we have $\mu(A) = \mu^*(A)$ by monotonicity of μ and the definition of μ^*. So for $A \in \mathcal{S}$, $\overline{\mu}(A) = \mu^*(A) = \mu(A)$. Therefore, $\overline{\mu}$ is an extension of μ. □

Proposition 17.13

Proposition 17.13. Let $\mathcal{S}$ be a semiring of subsets of a set X. Define $\mathcal{S}'$ to be the collection of unions of finite disjoint collections of sets in $\mathcal{S}$. Then $\mathcal{S}'$ is closed with respect to the formation of relative complements. Furthermore, any premeasure on $\mathcal{S}$ has a unique extension to a premeasure on $\mathcal{S}'$.

Proof. (1) Since $\mathcal{S}'$ consists of all unions of finite disjoint collections of sets in $\mathcal{S}$, then an element of $\mathcal{S}'$ is of the form $\cup_{k=1}^{n} S_k$ where $S_k \in \mathcal{S}$. So the union of two sets in $\mathcal{S}'$ is of the form $\cup_{k=1}^{n} A_n) \cup (\cup_{j=1}^{m} B_j)$ where each $A_k, B_j \in \mathcal{S}$. If some A_k intersects some B_j then $A_k \cap B_j \in \mathcal{S}$, since $\mathcal{S}$ is a semiring, and $A_k \setminus B_j$ and $B_j \setminus A_k$ are each unions of disjoint elements of $\mathcal{S}$. It follows that $\mathcal{S}'$ is closed under finite unions. Similarly, $(\cup_{k=1}^{n} A_k) \cap (\cup_{j=1}^{m} B_j)$ can be expressed as a union of disjoint elements of $\mathcal{S}$, and $\mathcal{S}'$ is closed under finite intersections.

Proposition 17.13 (continued 1)

Proof (continued). With the above notation, $(\cup_{k=1}^{n} A_k) \setminus (\cup_{j=1}^{m} B_j) = \cup_{k=1}^{n} (\cap_{j=1}^{m} (A_k \setminus B_j))$. Now each $A_k \setminus B_j$ is a union of a finite union of disjoint elements of $\mathcal{S}$ since $\mathcal{S}$ is a semiring, and so $\mathcal{S}'$ is closed with respect to relative complements.

(2) Let $\mu : \mathcal{S} \to [0, \infty]$ be a premeasure on $\mathcal{S}$. For $E \subset X$ such that $E = \cup_{k=1}^{n} A_k \in \mathcal{S}'$ where the A_k are disjoint elements of $\mathcal{S}$, define $\mu'(E) = \sum_{k=1}^{n} \mu(A_k)$. Since we have defined $\mu'(E)$ in terms of a representation of E as a union of disjoint elements of $\mathcal{S}$, we need to verify that $\mu'(E)$ is independent of the representation of E as such a union (i.e., we need to make sure $\mu(E)$ is well defined). Suppose $E = \cup_{j=1}^{m} B_j$ where the B_j are disjoint elements of $\mathcal{S}$. Then $\mu(B_j) = \sum_{j=1}^{n} \mu(B_j \cap A_k)$ for $1 \leq j \leq m$ (by finite additivity of premeasure μ), and $\mu(A_k) = \sum_{j=1}^{m} \mu(B_j \cap A_k)$ for $1 \leq k \leq n$ (by the finite additivity of premeasure μ).

Proposition 17.13 (continued 2)

Proof (continued). Therefore

$$\sum_{j=1}^{m} \mu(B_j) = \sum_{j=1}^{m} \left(\sum_{k=1}^{n} \mu(B_j \cap A_k) \right) = \sum_{k=1}^{n} \left(\sum_{j=1}^{m} \mu(B_j \cap A_k) \right) = \sum_{k=1}^{n} \mu(A_k).$$

Thus μ' is properly defined on $\mathcal{S}'$ and uniquely determined by μ. We now need to show that μ' is a premeasure on $\mathcal{S}'$. Since μ is finitely additive, then μ' inherits finite additivity from μ. For countable monotonicity of μ', let $E \in \mathcal{S}'$ be covered by $\{E_K\}_{k=1}^{\infty}$ of set in $\mathcal{S}'$. By Problem 17.31(iii), we may assume the E_k are disjoint. Since $E \in \mathcal{S}$, $E = \cup_{j=1}^{m} A_j$ where the $A_j \in \mathcal{S}$ are disjoint (by the definition of $\mathcal{S}'$). For each $1 \leq j \leq m$, A_j is covered by $\cup_{k=1}^{\infty}(A_j \cap E_k)$, which is a countable collection of sets in $\mathcal{S}$ (since $\mathcal{S}$ is a semiring and so closed under intersections), and so by countable monotonicity of μ,

$$\mu(A_j) \leq \sum_{k=1}^{\infty} \mu(A_j \cap E_k). \qquad (*)$$

Proposition 17.13 (continued 3)

Proof (continued). Next,

$$\begin{aligned}
\nu'(E) &= \mu'\left(\cup_{j=1}^{m} A_j\right) = \mu\left(\cup_{j=1}^{m} A_j\right) \text{ since } A_j \in \mathcal{S} \\
&= \sum_{j=1}^{m} \mu(A_j) \text{ by the finite additivity of } \mu \\
&\leq \sum_{j=1}^{m} \left(\sum_{k=1}^{\infty} \mu(A_j \cap E_k) \right) \text{ by } (*) \\
&= \sum_{k=1}^{\infty} \left(\sum_{j=1}^{m} \mu(A_j \cap E_k) \right) \\
&= \sum_{k=1}^{\infty} \mu(E \cap E_k) \text{ since } E = \cup_{j=1}^{m} A_j \text{ and finite additivity of } \mu
\end{aligned}$$

Proposition 17.13 (continued 4)

Proposition 17.13. Let $\mathcal{S}$ be a semiring of subsets of a set X. Define $\mathcal{S}'$ to be the collection of unions of finite disjoint collections of sets in $\mathcal{S}$. Then $\mathcal{S}'$ is closed with respect to the formation of relative complements. Furthermore, any premeasure on $\mathcal{S}$ has a unique extension to a premeasure on $\mathcal{S}'$.

Proof (continued).

$$\begin{aligned}
\mu'(E) &\leq \sum_{k=1}^{\infty} \mu(E_k) \text{ by monotonicity of } \mu \\
&= \sum_{k=1}^{\infty} \mu'(E_k) \text{ since } \mu = \mu' \text{ on } \mathcal{S} \text{ and } E_k \in \mathcal{S}.
\end{aligned}$$

So μ' is countably monotone.
Finally, countable monotonicity of μ' implies that if $\varnothing \in \mathcal{S}'$ then $\mu'(\varnothing) = 0$. So μ' is a premeasure on $\mathcal{S}'$. □

The Carathéodory-Hahn Theorem

The Carathéodory-Hahn Theorem.
Let $\mu : \mathcal{S} \to [0, \infty]$ be a premeasure on a semiring $\mathcal{S}$ of subsets of X. Then the Carathéodory measure $\overline{\mu}$ induced by μ is an extension of μ. Furthermore, if μ is σ-finite, then so is $\overline{\mu}$ and $\overline{\mu}$ is the unique measure on the σ-algebra of μ^*-measurable sets that extends μ.

Proof. (This is much more detailed than the text's proof.) Define $\mathcal{S}'$ to be the collection of unions of finite disjoint collections of sets in $\mathcal{S}$. Then premeasure μ on $\mathcal{S}$ has a unique extension to a premeasure on $\mathcal{S}'$ by Proposition 17.13. Also by Proposition 17.13, $\mathcal{S}'$ is closed with respect to the formation of relative complements. By Theorem 17.12, the Carathéodory extension $\overline{\mu}$ of μ from $\mathcal{S}'$ to $\mathcal{M}$ (the σ-algebra of measurable sets) is in fact an extension of μ on $\mathcal{S}'$ (and therefore an extension of μ on $\mathcal{S}$).

The Carathéodory-Hahn Theorem (continued 1)

Proof (continued). Now suppose μ is σ-finite. Then $X = \cup_{k=1}^{\infty} S_k$ where $S_k \in \mathcal{S}$ and $\mu(S_k) < \infty$ for all $k \in \mathbb{N}$. So $\overline{\mu}(S_k) < \infty$ since $\overline{\mu}$ extends μ, and so $\overline{\mu}$ is σ-finite. For uniqueness, suppose μ_1 is another measure on $\mathcal{M}$ that extends μ. We express $X = \uplus_{k=1}^{\infty} X_k$ where the X_k are disjoint, $X_k \in \mathcal{S}'$ and $\mu(X_k) < \infty$ (this can be done with disjoint sets since $\mathcal{S}'$ is closed under relative complements). Since a measure (here, $\overline{\mu}$ and μ_1) is countably additive (as is required by the definition of *measure*), to prove uniqueness it suffices to show that $\overline{\mu}$ and μ_1 agree on the measurable subsets in each X_k (since any subset of X can be written as a countable union of such sets: $A \subset X$ satisfies $A = \uplus(A \cap X_k)$). Let E be measurable, $E \in \mathcal{M}$, and $E \subset E_0$ where $E_0 \in \mathcal{S}'$ and $\mu(E_0) < \infty$. We need to show that $\overline{\mu}(E) = \mu_1(E)$.

The Carathéodory-Hahn Theorem (continued 2)

Proof (continued). By Proposition 17.10, there is $A \in \mathcal{S}_{\sigma\delta}$ for which $E \subset A$ and $\overline{\mu}(A \setminus E) = 0$. We may assume $A \subset E_0$ (otherwise we replace A with $A \cap E_0 \in \mathcal{S}_{\sigma\delta}$); notice that E_0 is a finite union of disjoint sets in $\mathcal{S}$ so $E_0 \in \mathcal{S}_{\sigma\delta}$. Now if B is measurable and $\mu^*(B) = 0$ (and so $\overline{\mu}(B) = 0$), then $\mu^*(B) = \inf\{\sum \mu(E_k)\}$ where the infimum is taken over all coverings of B of the form $\{E_k\} \subset \mathcal{S}$. Since μ_1 extends μ then the countable monotonicity of μ_1 (a property of measure μ_1 by Proposition 17.1), $\mu_1(E) \le \sum \mu_1(E_k) = \sum \mu(E_k)$ for any cover $\{E_k\} \subset \mathcal{S}$ of B, and so $\mu_1(B) = 0$. Therefore (considering the $\overline{\mu}$-measure zero set $B = A \setminus E$), $\mu_1(A \setminus E) = 0$. By the countable additivity of μ_1 and $\overline{\mu}$, these measures agree on $\mathcal{S}_\sigma$ (since $\mathcal{S}$ is a semiring, each element of $\mathcal{S}_\sigma$ can be written as a countable *disjoint* union of elements of $\mathcal{S}$). Now for any $S \in \mathcal{S}_{\sigma\delta}$ with $S \subset E_0$, we have $S = \cap_{k=1}^{\infty} S_k$ for some $S_k \in \mathcal{S}_\sigma$.

The Carathéodory-Hahn Theorem (continued 3)

The Carathéodory-Hahn Theorem.
Let $\mu : \mathcal{S} \to [0, \infty]$ be a premeasure on a semiring $\mathcal{S}$ of subsets of X. Then the Carathéodory measure $\overline{\mu}$ induced by μ is an extension of μ. Furthermore, if μ is σ-finite, then so is $\overline{\mu}$ and $\overline{\mu}$ is the unique measure on the σ-algebra of μ^*-measurable sets that extends μ.

Proof (continued). Then define $D_n = \cap_{k=1}^{n} S_k$. Then each $D_n \in \mathcal{S}_\sigma$ (since $\mathcal{S}$ is a semiring) and $\{D_n\}$ is a descending sequence with $\lim D_n = S$. So, by continuity of measure (Proposition 17.2), since $\mu(E_0) < \infty$,

$$\overline{\mu}(S) = \overline{\mu}(\lim D_n) = \lim \overline{\mu}(D_n) = \lim \mu_1(D_n) = \mu_1(\lim D_n) = \mu_1(S).$$

So μ_1 and $\overline{\mu}$ agree on $\mathcal{S}_{\sigma\delta}$ subsets of E_0. Therefore $\mu_1(A) = \overline{\mu}(A)$. Hence $\mu_1(A \setminus E) = \overline{\mu}(A \setminus E) = 0$ implies by the excision principle (Proposition 17.1) that $\mu_1(A) - \mu_1(E) = \overline{\mu}(A) - \overline{\mu}(E)$ (since $E \subset A \subset E_0$ and $\mu(E_0) < \infty$) and so $\mu_1(E) = \overline{\mu}(E)$. It follows that μ_1 and $\overline{\mu}$ are equal on the σ-algebra of μ^*-measurable sets, and so $\overline{\mu}$ is unique. □

Corollary 17.14

Corollary 17.14. Let $\mathcal{S}$ be a semiring of subsets of a set X and $\mathcal{B}$ the smallest σ-algebra of subsets of X that contain $\mathcal{S}$. Then two σ-finite measures on $\mathcal{B}$ are equal if and only if they agree on sets in $\mathcal{S}$.

Proof. Let μ_1 and μ_2 be σ-finite measures on $\mathcal{B}$. First, if μ_1 and μ_2 do not agree on $\mathcal{S}$, then they are not equal on $\mathcal{B}$ (since $\mathcal{S} \subset \mathcal{B}$). Second, if μ_1 and μ_2 are σ-finite measures on $\mathcal{B}$, then their restrictions to $\mathcal{S}$ are σ-finite, finite additive (by the definition of measure), and countably monotone (by Proposition 17.1). So the restrictions of μ_1 and μ_2 to $\mathcal{S}$ are σ-finite premeasures which agree on $\mathcal{S}$. Therefore, by the Carathédory-Hahn Theorem, their extensions to $\mathcal{B}$ (a sub-σ-algebra of $\mathcal{M}$) are unique. That is, $\mu_1 = \mu_2$ on $\mathcal{B}$. □