Real Analysis

Chapter 18. Integration Over General Measure Spaces

18.1. Measurable Functions—Proofs of Theorems

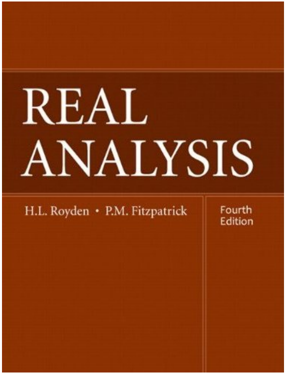

Table of contents

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Proposition 18.3. Let $(X, \mathcal{M}, \mu)$ be a complete measure space and X_0 a measurable subset of X for which $\mu(X \setminus X_0) = 0$. Then an extended real-valued function f on X is measurable if and only if its restriction to X_0 is measurable. In particular, if g and h are extended real-valued functions on X for which $g = h$ a.e. on X, then g is measurable if and only if h is measurable.

Proof. Define f_0 to be the restriction of f to X_0. Let $c \in \mathbb{R}$ and $E = (c, \infty)$. If f is measurable, then $f^{-1}(E)$ is measurable and therefore so is $f^{-1}(E) \cap X_0 = f_0^{-1}(E)$. So f_0 is measurable.

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Now assume f_0 is measurable. Then $f^{-1}(E) = f_0^{-1}(E) \cup A$ where A is a subset of $X \setminus X_0$. Since $(X, \mathcal{M}, \mu)$ is complete, A is measurable and therefore $f^{-1}(E) = f_0^{-1}(E) \cup A$ is measurable. So f is measurable.

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Theorem 18.5

Proposition 18.5. Let $(X, \mathcal{M})$ be a measurable space, f a measurable real-valued function on X, and $\varphi : \mathbb{R} \to \mathbb{R}$ continuous. Then the composition $\varphi \circ f : X \to \mathbb{R}$ also is measurable.

Proof. Let $\mathcal{O}$ be an open set of real numbers. Since $\varphi : \mathbb{R} \to \mathbb{R}$ is continuous, $\varphi^{-1}(\mathcal{O})$ is open. By Proposition 18.2, $f^{-1}(\varphi^{-1}(\mathcal{O})) = (\varphi \circ f)^{-1}(\mathcal{O})$ is a measurable set since f is measurable, and so $\varphi \circ f$ is a measurable function. □

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Theorem 18.6. Let $(X, \mathcal{M}, \mu)$ be a measure space and $\{f_n\}$ a sequence of measurable functions on X for which $\{f_n\} \to f$ pointwise a.e. on X. If either the measure space $(X, \mathcal{M}, \mu)$ is complete or the convergence is pointwise on all of X, then f is measurable.

Proof. By Proposition 18.3, possibly be excising a set of measure 0 from X, without loss of generality we can assume that $\{f_n\}$ converges pointwise on all of X (completeness is needed here). Let $c \in \mathbb{R}$ be finite.

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$$\{x \in X \mid f(x) < c\} = \cup_{k,n\in\mathbb{N}} \left(\cap_{j=k}^{\infty}\{x \in X \mid f_j(x) < c - 1/n\}\right)$$

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The Simple Approximation Lemma

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Let $(X, \mathcal{M})$ be a measurable space and f a measurable function on X that is bounded on X. Then for each $\epsilon > 0$, there are simple functions ϕ_ϵ and ψ_ϵ on X such that $\phi_\epsilon \leq f \leq \psi_\epsilon$ and $0 \leq \psi_\epsilon - \phi_\epsilon < \epsilon$ on X.

Proof. Since f is bounded, there is $[c, d)$ such that $[c, d) \supset f(X)$. Let $c = y_0 < y_1 < \cdots < y_n = d$ be a partition of $[c, d]$ such that $y_k - y_{k-1} < \varepsilon$ for $q \leq k \leq n$.

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$$\varphi_\varepsilon = \sum_{k=1}^{n} y_{k-1} \chi_{X_k} \text{ and } \psi_\varepsilon = \sum_{k=1}^{n} y_k \chi_{X_k}.$$

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Let $(X, \mathcal{M}, \mu)$ be a measure space and f a measurable function on X. Then there is a sequence $\{\psi_n\}$ of simple functions on X that converges pointwise on X to f and $|\psi_n| \leq |f|$ on X for all $n \in \mathbb{N}$.

(i) If X is σ-finite, then we may choose the sequence $\{\psi_n\}$ so that each ψ_ϵ vanishes outside a set of finite measure.

(ii) If f is nonnegative, we may choose the sequence $\{\psi_n\}$ to be increasing and each $\psi_n \geq 0$ on X.

Proof. Fix $n \in \mathbb{N}$ and define $E_n = \{x \in X \mid |f(x)| \leq n\}$. Since $|f|$ is a measurable function, then E_n is a measurable set. The restriction of f to E_n is measurable and bounded by $-n$ and n.

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The Simple Approximation Theorem (continued 1)

Proof (continued). For $x \in E_n$ define

$$\psi_n(x) = \begin{cases} 0 & \text{if } f(x) = 0 \\ \max\{h_n(x), 0\} & \text{if } f(x) > 0 \\ \min\{g_n(x), 0\} & \text{if } f(x) < 0. \end{cases}$$

Extend ψ_n to all of X by defining $\psi_n(x) = n$ if $f(x) > n$ and $\psi_n(x) = -n$ if $f(x) < -n$. By construction, $|\psi_n| \le |f|$ for all n. If $f(x)$ is finite, then there is $N \in \mathbb{N}$ such that $|f(x)| < N$. Then for $n \ge N$, $0 \le f(x) - \psi_n(x) \le g_n(x) - h_n(x) < 1/n$ and so $\lim_{n\to\infty} \psi_n(x) = f(x)$. If $|f(x)| = \infty$, then $|\varphi_n(x)| = n$ (and the sign of $\varphi_n(x)$ is the same as the sign of $f(x)$), and $\lim_{n\to\infty} \varphi_n(x) = f(x)$. So $\{\varphi_n\}$ converges to f pointwise on X.

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If f is nonnegative, replace psi_n by $\max_{1 \leq i \leq n}\{\psi_i\}$, which is measurable by Corollary 18.7 and is simple. Also, $\{\psi_n\}$ is an increasing sequence of nonnegative functions, so (ii) holds. □

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