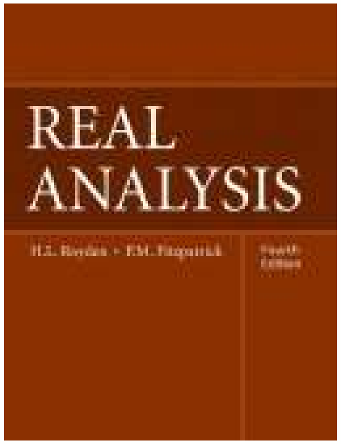
Real Analysis

Chapter 18. Integration Over General Measure Spaces

18.2. Integration of Nonnegative Measurable Functions—Proofs of Theorems



Proposition 18.8

Proposition 18.8. Let $(X, \mathcal{M}, \mu)$ be a measure space and let φ and ψ be nonnegative simple functions on X. If α and β are positive real numbers, then

$$\int_X (\alpha\psi + \beta\varphi)\, d\mu = \alpha \int_X \psi\, d\mu + \beta \int_X \varphi\, d\mu. \quad (2)$$

If A and B are disjoint measurable subsets of X, then

$$\int_{A\cup B} \psi\, d\mu = \int_A \psi\, d\mu + \int_B \psi\, d\mu. \quad (3)$$

In particular, if $X_0 \subset X$ is measurable and $\mu(X \setminus X_0) = 0$, then

$$\int_X \psi\, d\mu = \int_{X_0} \psi\, d\mu. \quad (4)$$

Furthermore, if $\psi \le \varphi$ a.e. on X, then

$$\int_X \psi\, d\mu \le \int_X \varphi\, d\mu. \quad (5)$$

Proposition 18.8 (continued 1)

Proof. If either ψ or φ is positive on a set of infinite measure, then the linear combination $\alpha\psi + \beta\varphi$ is also positive on a set of infinite measure (and simple), and so both sides of (2) are ∞ and (2) holds. Next, suppose both ψ and φ vanish outside a set of finite measure, and hence so does $\alpha\psi + \beta\varphi$. The proof of linearity, (2), now follows exactly as it did in Chapter 4 (in the proofs of Lemma 4.1 and Proposition 4.2, see page 72).

For disjoint A and B, we have $\psi\chi_{A\cup B} = \psi\chi_A + \psi\chi_B$ on X, and additivity over domains, (3), then follows from (2).

Next, (4) follows from (3) since the integral of a since the integral of a simple function over a set of measure zero is zero by definition.

Proposition 18.8 (continued 2)

Proof. For monotonicity of the integral, (5), we may assume from (4) that $\psi \le \varphi$ on all of X. Since φ and ψ are simple and so only take on a finite number of values, then we can write $X = \cup_{k=1}^n X_k$ where the X_k are disjoint and measurable and $\psi = \sum_{k=1}^n a_k \chi_{X_k}$ an $d\varphi = \sum_{k=1}^n b_k \chi_{X_k}$ where $0 \le a_k \le b_k$ for $1 \le k \le n$. Then by linearity, (2), and the definition of integral,

$$\int_X \psi\, d\mu = \sum_{k=1}^n a_k \mu)(X_k) \le \sum_{k=1}^n b_k \mu(X_k) = \int_X \varphi\, d\mu,$$

and so (5) follows. □

Chebyshev's Inequality

Chebyshev's Inequality.
Let $(X, \mathcal{M}, \mu)$ be a measure space, f a nonnegative measurable function on X, and $\lambda > 0$ a real number. Then

$$\mu\{x \in X \mid f(x) > \lambda\} \le \frac{1}{\lambda}\int_X f\,d\mu.$$

Proof. Define $X_\lambda = \{x \in X \mid f(x) \ge \lambda\}$ and $\varphi = \lambda\chi_{X_\lambda}$. Then φ is simple and $0 \le \varphi \le f$ on X. So by definition (and by Lemma) $\lambda\mu(X_\lambda) = \int_X \varphi\,d\mu \le \int_X f\,d\mu$. The result now follows. □

Proposition 18.9

Proposition 18.9. Let $(X, \mathcal{M})$ be a measure space and f nonnegative measurable function on X for which $\int_X f\,d\mu < \infty$. Then f is finite a.e. on X and $\{x \in X \mid f(x) > 0\}$ is σ-finite.

Proof. Define $X_\infty = \{x \in X \mid f(x) = \infty\}$ and $X_n = \{x \in X \mid f(x) \ge n\}$. Then $X_\infty \subset X_n$ for all $n \in \mathbb{N}$ and by monotonicity of measure (Proposition 17.1) $\mu(X_\infty) \le \mu(X_n)$ for all $n \in \mathbb{N}$. By Chebyshev's Inequality with $\lambda = n$, $\mu(X_n) \le \frac{1}{n}\int_X f\,d\mu < \infty$ for $n \in \mathbb{N}$, and so $\mu(X_\infty) \le \mu(X_n) \le \frac{1}{n}\int_X f\,d\mu$ for all $n \in \mathbb{N}$. Since $\int_X f\,d\mu$ is finite, $\mu(X_\infty) = 0$. (This is the same as the proof of Chebyshev's Inequality from Chapter 4; see page 84.)

Now define $X_n = \{x \in X \mid f(x) \ge 1/n\}$. Then by Chebyshev's Inequality with $\lambda = 1/n$, $\mu(X_n) \le n\int_X f\,d\mu < \infty$, and each $\mu(X_n)$ is of finite measure. So $\{x \in X \mid f(x) > 0\} = \cup_{n=1}^\infty E_n$, and $\{x \in X \mid f(x) > 0\}$ is a countable union of finite measurable sets. That is, it is σ-finite. □

Fatou's Lemma

Fatou's Lemma.
Let $(X, \mathcal{M}, \mu)$ be a measure space and $\{f_n\}$ be a sequence of nonnegative measurable functions on X where $\{f_n\} \to f$ a.e. on X. Assume f is measurable. Then

$$\int_X f\,d\mu \le \liminf \int_X f_n\,d\mu.$$

Proof. Let X_0 be a measurable subset of X for which $\mu(X \setminus X_0) = 0$ and $\{f_n\} \to f$ pointwise on X_0. By (9) of "Lemma," each side of the claim of Fatou's Lemma remains unchanged if X is replaced by X_0. So, without loss of generality, $\{f_n\} \to f$ pointwise on X. Since $\int_X f\,d\mu$ is defined in terms of simple functions φ for which $0 \le \varphi \le f$, if we establish that

$$\int_X \varphi\,d\mu \le \liminf \int_X f_n\,d\mu \qquad (12)$$

for all such φ, then Fatou's lemma will follow by taking a supremum over such φ. So let φ be simple with $0 \le \varphi \le f$ on X.

Fatou's Lemma (continued 1)

Proof (continued). If $\int_X \varphi\,d\mu = 0$ then the desired inequality holds, so assume $\int_X \varphi\,d\mu > 0$.

Case 1. Suppose $\int_X \varphi\,d\mu = \infty$. Then there is a measurable $X_\infty \subset X$ and $a > 0$ for which $\mu(X_\infty) = \infty$ and $\varphi = a$ on X_∞.

For each $n \in \mathbb{N}$, define $A_n = \{x \in X \mid f_k(x) \ge a/2 \text{ for all } k \ge n\}$. Then $\{A_n\}$ is an ascending sequence of measurable sets ($A_n \subset A_{n+1}$; larger n implies fewer f_k and so more x values). Since $X_\infty \subset \cup_{n=1}^\infty A_n$ then by Continuity of Measure (Proposition 17.2) and Monotonicity of Measure (Proposition 17.1),

$$\lim_{n\to\infty} \mu(A_n) = \mu\left(\lim_{n\to\infty} A_n\right) = \mu\left(\cup_{n=1}^\infty A_n\right) \ge \mu(X_\infty) = \infty.$$

But by Chebyshev's Inequality, for each $n \in \mathbb{N}$,

$$\mu(A_n) \le \frac{2}{a}\int_{A_n} f_n\,d\mu \le \frac{2}{a}\int_X f_n\,d\mu.$$

Fatou's Lemma (continued 2)

Proof (continued). So as $n \to \infty$, the left hand side approaches ∞ and so the right hand side approaches ∞ and

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \infty = \int_X \varphi \, d\mu.$$

Case 2. Suppose $0 < \int_X \varphi \, d\mu < \infty$. Now $\varphi = 0$ on some subset of X, say X_0. Then $\int_X \varphi \, d\mu = \int_{X_0} \varphi \, d\mu$. Also, $\liminf \int_{X_0} f_n \, d\mu \leq \liminf \int_X f_n \, d\mu$. So if we can verify that $\int_{X_0} \varphi \, d\mu \leq \liminf \int_{X_0} \varphi_n \, d\mu$, then (12) will follow. That is, without loss of generality we can assume $\varphi > 0$ on all of X. Since we are hypothesizing that $\int_X \varphi \, d\mu < \infty$, this implies that $\mu(X) < \infty$. Let $\varepsilon > 0$ and define

$$X_n = \{x \in X \mid f_k(x) > (1-\varepsilon)\varphi(x) \text{ for all } k \geq n\}.$$

Then $\{X_n\}$ is an ascending sequence of measurable subsets of X (similar to Case 1) whose union equals X (since $\varphi \leq f$). So $\{X \setminus X_n\}$ is a descending sequence of measurable sets whose intersection is $\varnothing$.

Fatou's Lemma (continued 3)

Proof (continued). Since $\mu(X) < \infty$, by the Continuity of Measure (Proposition 17.2) $\lim_{n\to\infty} \mu(X \setminus X_n) = 0$. So choose $N \in \mathbb{N}$ such that $\mu(X \setminus X_n) < \varepsilon$ for all $n \geq N$. Define $M > 0$ to be the maximum of the values taken on by simple φ on X. Then

$$\begin{aligned}
\int_X f_n \, d\mu &\geq \int_{X_n} f_n \, d\mu \text{ by Monotonicity ((8) of "Lemma")} \\
&\geq (1-\varepsilon)\int_{X_n} \varphi \, d\mu \text{ by definition of } X_n \\
&= (1-\varepsilon)\int_X \varphi \, d\mu - (1-\varepsilon)\int_{X\setminus X_n} \varphi \, d\mu \text{ by Additivity for} \\
&\quad\ \text{nonnegative simple functions ((3) of Proposition 18.8)} \\
&\quad\ \text{and the finiteness of the integrals of } \varphi \\
&\geq (1-\varepsilon)\int_X \varphi \, d\mu - \int_{X\setminus X_n} \varphi \, d\mu
\end{aligned}$$

Fatou's Lemma (continued 4)

Proof (continued).

$$\begin{aligned}
\int_X f_n \, d\mu &\geq (1-\varepsilon)\int_X \varphi \, d\mu - \varepsilon M \text{ by (5) of Proposition 18.8} \\
&= \int_X \varphi \, d\mu - \varepsilon\left(\int_X \varphi \, d\mu + M\right).
\end{aligned}$$

So

$$\liminf \int_X f_n \, d\mu \geq \int_X \varphi \, d\mu - \varepsilon\left(\int_X \varphi \, d\mu + M\right)$$

and since $\varepsilon > 0$ is arbitrary and $\int_X \varphi \, d\mu + M$ is finite, then (12) follows. □

The Monotone Convergence Theorem

The Monotone Convergence Theorem.
Let $(X, \mathcal{M}, \mu)$ be a measure space and $\{f_n\}$ an increasing sequence (i.e., pointwise increasing) of nonnegative measurable functions on X. Define $f(x) = \lim_{n\to\infty} f_n(x)$ for each $x \in X$. Then

$$\lim_{n\to\infty}\left(\int_X f_n \, d\mu\right) = \int_X \left(\lim_{n\to\infty} f_n\right) d\mu = \int_X f \, d\mu.$$

Proof. By Theorem 18.6, f is measurable. By Fatou's Lemma, $\int_X f \leq \liminf \int_X f_n \, d\mu$. Since $f_n \leq f$ on X, by (18) of "Lemma" we have $\int_X f_n \, d\mu \leq \int_X f \, d\mu$ for all $n \in \mathbb{N}$. So

$$\limsup \int_X f_n \, d\mu \leq \int_X f \, d\mu,$$

and the result follows. □

Beppo Levi's Lemma

Beppo Levi's Lemma.
Let $(X, \mathcal{M}, \mu)$ be a measure space and $\{f_n\}$ an increasing sequence of nonnegative measurable functions on X. If the sequence of integrals $\{\int_X f_n \, d\mu\}$ is bounded, then $\{f_n\}$ converges pointwise on X to a measurable function f that is finite a.e. on X and

$$\lim_{n\to\infty} \left(\int_X f_n \, d\mu \right) = \int_X \left(\lim_{n\to\infty} f_n \right) \, d\mu = \int_X f \, d\mu < \infty.$$

Proof. Define $f(x) = \lim_{n\to\infty} f_n(x)$ for each $x \in X$ (since $\{f_n\}$ is increasing, the pointwise limits exist as extended real numbers). Since $\{f_n\}$ is increasing, by the Monotone Convergence Theorem $\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu$. Since the sequence $\{\int_X f_n \, d\mu\}$ is bounded, then $\lim_{n\to\infty} \int_X f_n \, d\mu$ is bounded; that is, $\int_X f \, d\mu < \infty$. by proposition 18.9, f is finite a.e. on X. □

Proposition 18.10

Proposition 18.10. Let $(X, \mathcal{M}, \mu)$ be a measure space and f a nonnegative measurable function on X. Then there is an increasing sequence $\{\psi_n\}$ of simple functions on X that converges pointwise on X to f and

$$\lim_{n\to\infty} \left(\int_X \psi_n \, d\mu \right) = \int_X f \, d\mu.$$

Proof. By the Simple Approximation Theorem part (ii), there is an increasing sequence $\{\varphi_n\}$ of simple functions which converges pointwise to f on X. By the Monotone Convergence Theorem,

$$\lim_{n\to\infty} \int_X \varphi_n \, d\mu = \int_X \lim_{n\to\infty} \varphi_n \, d\mu = \int_X f \, d\mu.$$

□

Proposition 18.11

Proposition 18.11. Linearity of Integrals of Nonnegative Measurable Functions.
Let $(X, \mathcal{M}, \mu)$ be a measure space and f and g nonnegative measurable functions on X. If α and β are positive real numbers, then

$$\int_X (\alpha f + \beta g) \, d\mu = \alpha \int_X f \, d\mu + \beta \int_X g \, d\mu.$$

Proof. By (7) of "Lemma" we have that $\int_X \alpha f \, d\mu = \alpha \int_X f \, d\mu$, so we need only establish the result for $\alpha = \beta = 1$ (i.e., for the "addition of functions" part). By Proposition 18.10, there are increasing sequences $\{\psi_n\}$ an d$\{\varphi_n\}$ of nonnegative simple functions that converge pointwise to g and f respectively for which

$$\lim_{n\to\infty} \int_X \psi_n \, d\mu = \int_X g \, d\mu \text{ and } \lim_{n\to\infty} \int_X \varphi_n \, d\mu = \int_X f \, d\mu.$$

Proposition 18.11 (continued)

Proof (continued). Then $\{\varphi_n + \psi_n\}$ is an increasing sequence of simple functions that converges pointwise to $f + g$. Then

$$\begin{aligned}
\int_X (f+g) &= \lim_{n\to\infty} \int_X (\varphi_n + \psi_n) \, d\mu \text{ by the Monotone Convergence Theorem} \\
&= \lim_{n\to\infty} \left(\int_X \varphi_n \, d\mu + \int_X \varphi_n \, d\mu \right) \text{ by proposition 18.8 (2)} \\
&= \lim_{n\to\infty} \int_X \varphi_n \, d\mu + \lim_{n\to\infty} \int_X \psi_n \, d\mu \\
&= \int_X f \, d\mu + \int_X g \, d\mu.
\end{aligned}$$

□