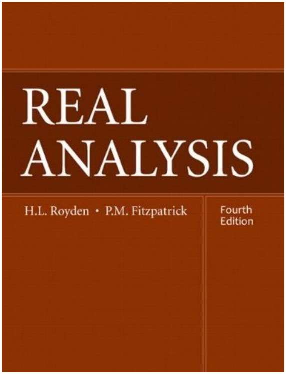
Real Analysis

Chapter 18. Integration Over General Measure Spaces

18.3. Integration of General Measurable Functions—Proofs of Theorems



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Let $(X, \mathcal{M}, \mu)$ be a measure space and f a measurable function on X. If g is integrable over X and dominates f on X in the sense that $|f| \le g$ a.e. on X, then f is integrable over X and

$$\left|\int_X f\, d\mu\right| \le \int_X |f|\, d\mu \le \int_X g\, d\mu.$$

Proof. Since $|f| \le g$ a.e. on X, (8) of "Lemma" implies that $|f|$ is integrable over X. Then

$$\begin{aligned}\left|\int_X f\, d\mu\right| &= \left|\int_X f^+\, d\mu - \int_X f^-\, d\mu\right| \text{ by linearity (Proposition 18.11)}\\ &\le \int_X f^+\, d\mu + \int_X f^-\, d\mu \text{ by the Triangle Inequality}\\ &= \int_X |f|\, d\mu \le \int_X f\, d\mu \text{ by (8) of Lemma.}\end{aligned}$$

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Proof. Since $|f| \leq g$ a.e. on X, (8) of "Lemma" implies that $|f|$ is integrable over X. Then

$$\begin{aligned} \left| \int_X f \, d\mu \right| &= \left| \int_X f^+ \, d\mu - \int_X f^- \, d\mu \right| \text{ by linearity (Proposition 18.11)} \\ &\leq \int_X f^+ \, d\mu + \int_X f^- \, d\mu \text{ by the Triangle Inequality} \\ &= \int_X |f| \, d\mu \leq \int_X f \, d\mu \text{ by (8) of Lemma.} \end{aligned}$$

□

Theorem 18.12

Theorem 18.12. Let $(X, \mathcal{M}, \mu)$ be a measure space and let f and g be integrable over X.
(Linearity) For $\alpha, \beta \in \mathbb{R}$, $\alpha f + \beta g$ is integrable over X and

$$\int_X (\alpha f + \beta g)\, d\mu = \alpha \int_X f\, d\mu + \beta \int_X g\, d\mu.$$

(Monotonicity) If $f \leq g$ a.e. on X, then

$$\int_X f\, d\mu \leq \int_X g\, d\mu.$$

(Additivity Over Domains) If A and B are disjoint measurable sets, then

$$\int_{A \cup B} f\, d\mu = \int_A f\, d\mu + \int_B f\, d\mu.$$

Theorem 18.12 (continued 1)

Proof. We deal with linearity in two steps. First, let $\alpha \in \mathbb{R}$, $\alpha > 0$, and consider αf. By definition of integral, we have

$$\begin{aligned}
\int_X \alpha f \, d\mu &= \int_X (\alpha r)^+ - \int_X (\alpha f)^- \, d\mu \\
&= \int_X \alpha f^+ \, d\mu - \int_X \alpha f^- \, d\mu \\
&= \alpha \int_X f^+ \, d\mu - \alpha \int_X f^- \, d\mu \text{ by Proposition 18.11} \\
&= \alpha \left(\int_X f^+ \, d\mu - \int_X f^- \, d\mu \right) = \alpha \int_X f \, d\mu.
\end{aligned}$$

Theorem 18.12 (continued 2)

Proof (continued). Similarly, for $\alpha < 0$,

$$\begin{aligned} \int_X \alpha f \, d\mu &= \int_X (\alpha f)^+ - \int_X (\alpha f)^- \, d\mu \\ &= \int_X (-\alpha) f^- \, d\mu - \int_X (-\alpha) f^+ \, d\mu \text{ since } (\alpha f)^+ = (-\alpha) f^- \\ &\qquad \text{and } (\alpha f)^- = (-\alpha) f^+ \\ &= -\alpha \int_X f^- \, d\mu - (-\alpha) \int_X f^+ \, d\mu \text{ by Proposition 18.11} \\ &= \alpha \left(\int_X f^+ \, d\mu - \int_X f^- \, d\mu \right) = \alpha \int_X f \, d\mu. \end{aligned}$$

Second, we consider $f + g$. By the definition of integrable, $|f|$ and $|g|$ are both integrable over X. So by Proposition 18.11, $|f| + |g|$ is also integrable over X. Since $|f + g| \leq |f| + |g|$ pointwise on X, by (8) of "Lemma" of the previous section we have that $|f + g|$ is integrable over X. So the positive and negative parts of f, g, and $f + g$ are integrable over X.

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$$\begin{aligned}
\int_X \alpha f \, d\mu &= \int_X (\alpha f)^+ - \int_X (\alpha f)^- \, d\mu \\
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Theorem 18.12 (continued 3)

Proof (contineud). By the note above, we may assume f and g are finite on all of X. Notice
$f+g=(f+g)^+-(f+g)^-=(f^+-f^-)+(g^+-g^-)$ pointwise on X, all of these are finite, and $(f+g)^++f^-+g^-=(f+g)^-+f^++g^+$ on X. By Proposition 18.11 (since all of these are nonnegative)

$$\int_X (f+g)^+\,d\mu+\int_X f^-\,d\mu+\int_X g^-\,d\mu$$
$$=\int_X (f+g)^-\,d\mu+\int_X f^+\,d\mu+\int_X g^+\,d\mu.$$

By the integrability of all of these, we can rearrange to get

$$\int_X (f+g)^+\,d\mu-\int_X (f+g)^-\,d\mu$$
$$=\int_X f^+\,d\mu-\int_X f^-\,d\mu+\int_X g^+\,d\mu-\int_X g^-\,d\mu$$

or $\int_X (f+g)\,d\mu=\int_X f\,d\mu+\int +Xg\,d\mu$. So we have linearity.

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Proof (continued). For monotonicity, assume $f \leq g$ a.e. on X. Then $g - f \geq 0$ a.e. on X and so by (8) of Lemma,

$$0 \leq \int_X (g - f) = \int_X g\, d\mu - \int_X f\, d\mu,$$

or $\int_X f\, d\mu \leq \int_X g\, d\mu$.

For additivity, with $A \cap B = \varnothing$ we have $f\chi_{A \cup B} = f(\chi_A + \chi_B)$ $= f\chi_A + f\chi_B$ on X, so

$$\begin{aligned} \int_X f\chi_{A \cup B}\, d\mu &= \int_{A \cup B} f\, d\mu \text{ by definition} \\ &= \int_X (fx\chi_A + f\chi_B)\, d\mu \text{ by substitution} \\ &= \int_X f\chi_A + \int_X \chi_B \text{ by linearity} \\ &= \int_A f + \int_B f \text{ by definition.} \end{aligned}$$

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Theorem 18.13. Countable Additivity

Theorem 18.13. Countable Additivity over Domains of Integration. Let $(X, \mathcal{M}, \mu)$ be a measure space, let function f be integrable over X, and let $\{X_n\}_{n=1}^{\infty}$ be a disjoint countable collection of measurable sets whose union is X. Then

$$\int_X f\,d\mu = \int_{\cup E_k} f\,d\mu = \sum_{n=1}^{\infty}\left(\int_{E_k} f\,d\mu\right).$$

Proof. We show the result for $f \geq 0$ on X, and the general result will then follow by considering positive and negative parts. For $n \in \mathbb{N}$, define $f_n = \sum_{k=1}^{n} f\chi_{X_n}$ on X.

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$$= \lim_{n\to\infty} \sum_{k=1}^{n} \left(\int_X f\chi_{X_n}\,d\mu \right) = \sum_{k=1}^{\infty} \left(\int_X f\,d\mu \right).$$

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Let $(X, \mathcal{M})$ be a measure space and let the function f be integrable over X.

(i) If $\{X_n\}_{n=1}^{\infty}$ is an ascending countable collection of measurable subsets of X whose union is X, then

$$\int_X f \, d\mu = \lim_{n\to\infty} \left(\int_{X_n} f \, d\mu \right).$$

(ii) If $\{X_n\}_{n=1}^{\infty}$ is a descending countable collection of measurable subsets of X, then

$$\int_{\cap X_n} f \, d\mu = \lim_{n\to\infty} \left(\int_{X_n} f \, d\mu \right).$$

Proof. Define $m : \mathcal{M} \to [0, \infty]$ as $m(E) = \int_E f \, d\mu$. By Countable Additivity over Domain, we have that m is countably additive. So m is a measure on $(X, \mathcal{M}, m)$ is a measure space. So, by the Continuity of Measure (Proposition 17.2), both (i) and (ii) follow. □

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Theorem 18.15

Theorem 18.15. Let $(X, \mathcal{M}, \mu)$ be a measure space and f a measurable function on X. If f is bounded on X and vanishes outside a set of finite measure, then f is integrable over X.

Proof. We show the result for $f \geq 0$ on X, and the general result will then follow by considering positive and negative parts. Let X_0 be a set of finite measure for which f vanishes on $X \setminus X_0$. Let $M \geq 0$ be a bound on f: $0 \leq f \leq M$ on X. Define $\varphi = M\chi_{X_0}$.

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The Lebesgue Dominated Convergence Theorem

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Let $(X, \mathcal{M}, \mu)$ be a measure space and let $\{f_n\}$ be a sequence of measurable functions on X for which $\{f_n\} \to f$ pointwise a.e. on X and suppose f is measurable. Assume there is a nonnegative function g that is integrable over X and dominates the sequence $\{f_n\}$ on X in the sense that $|f_n| \le g$ a.e. on X for all $n \in \mathbb{N}$. Then f is integrable over X and

$$\lim_{n\to\infty} \left(\int_X f_n \, d\mu \right) = \int_X \left(\lim_{n\to\infty} f_n \right) \, d\mu = \int_X f \, d\mu.$$

Proof. For each $n \in \mathbb{N}$, $g - f_n$ and $g + f_n$ are nonnegative measurable functions. By the Integral Comparison Test, for each $n \in \mathbb{N}$, f and f_n are integrable over X since g is integrable over X. So

$$\int_X g \, d\mu - \int_X f \, d\mu = \int_X (g - f) \, d\mu \text{ by linearity, Theorem 18.12}$$

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The Lebesgue Dominated Convergence Theorem (continued 1)

Proof (continued).

$$\begin{aligned}\int_X g\,d\mu - \int_X f\,d\mu &\le \liminf \int_X (g - f_n)\,d\mu \text{ by Fatou's Lemma}\\ &= \int_X g\,d\mu - \limsup \int_X f_n\,d\mu,\end{aligned}$$

and

$$\begin{aligned}\int_X g\,d\mu + \int_X f\,d\mu &= \int_X (g + f)\,d\mu \text{ by linearity, Theorem 18.12}\\ &\le \liminf \int_X (g + f_n)\,d\mu \text{ by Fatou's Lemma}\\ &= \int_X g\,d\mu + \liminf \int_X f_n\,d\mu.\end{aligned}$$

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Proof (continued). So

$$\begin{aligned} \limsup \int_X f_n \, d\mu &\leq \int_X f \, d\mu \text{ by the first inequality} \\ &\leq \liminf \int_X f \, d\mu \text{ by the second inequality.} \end{aligned}$$

Therefore,

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Proposition 18.17

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$$\text{if } \mu(E) < \delta \text{ then } \int_E |f| \, d\mu < \varepsilon. \tag{21}$$

Furthermore, for each $\varepsilon > 0$, there is a subset X_0 of X that has finite measure and

$$\int_{X \setminus X_0} |f| \, d\mu < \varepsilon.$$

Proof. We show the result for $f \geq 0$ on X, and the general result will follow by considering positive and negative parts. Let $\varepsilon > 0$. Since $\int_X f \, d\mu$ is finite, we have from the definition of the integral of a nonnegative function (page 367) that there is a simple function ψ on X for which $0 \leq \psi \leq f$ on X and $0 \leq \int_X f \, d\mu - \int_X \psi \, d\mu < \varepsilon/2$. Choose $M > 0$ such that $0 \leq \psi \leq M$ on X (which can be done since ψ is simple).

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Proposition 18.17 (continued 1)

Proof (continued). The in $E \subset X$,

$$\begin{aligned}
\int_E f\,d\mu &= \int_E (\psi + f - \psi)\,d\mu = \int_E \psi\,d\mu + \int_E (f-\psi)\,d\mu \text{ by linearity,} \\
&\qquad\qquad\qquad\qquad\text{Proposition 18.11} \\
&= \int_E \psi\,d\mu + \left(\int_E f\,d\mu - \int_E \psi\,d\mu\right) \text{ by linearity, Theorem 18.12} \\
&\le \int_E \psi\,d\mu + \frac{\varepsilon}{2} \\
&\le Mm(E) + \frac{\varepsilon}{2} \text{ by monotonicity, Proposition 18.8.}
\end{aligned}$$

If we take $\delta = \varepsilon/(2M)$, then (21) holds for any E with $m(E) < \delta$.

Proposition 18.17 (continued 2)

Proof (continued). Since ψ is simple and integrable over X, measurable set $X_0 = \{x \in X \mid \psi(x) > 0\}$ has finite measure. Moreover,

$$\begin{aligned} \int_{X \setminus X_0} f \, d\mu &= \int_{X \setminus X_0} (f - \psi) \, d\mu \\ &\leq \int_X (f - \psi) \, d\mu \text{ by additivity, Theorem 8.12} \\ &< \varepsilon. \end{aligned}$$

□

The Vitali Convergence Theorem

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Let $(X, \mathcal{M}, \mu)$ be a measure space and let $\{f_n\}$ be a sequence of functions on X that is both uniformly integrable and tight over X. Suppose $\{f_n\} \to f$ pointwise over X. Then

$$\lim_{n\to\infty}\left(\int_X f_n\,d\mu\right) = \int_X \left(\lim_{n\to\infty} f_n\right)\,d\mu = \int_X f\,d\mu.$$

Proof. For $n \in \mathbb{N}$, $|f - f_n| \le |f| + |f_n|$ pointwise on X. If X_0 and X_1 are measurable subsets of X for which $X_1 \subset X_0$, then $X = X_1 \dot\cup (X_0 \setminus X_1) \dot\cup (X \setminus X_0)$.

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$$\lim_{n\to\infty} \left(\int_X f_n \, d\mu \right) = \int_X \left(\lim_{n\to\infty} f_n \right) d\mu = \int_X f \, d\mu.$$

Proof. For $n \in \mathbb{N}$, $|f - f_n| \le |f| + |f_n|$ pointwise on X. If X_0 and X_1 are measurable subsets of X for which $X_1 \subset X_0$, then $X = X_1 \cup (X_0 \setminus X_1) \cup (X \setminus X_0)$. So for all $n \in \mathbb{N}$,

$$\begin{aligned} \left| \int_X (f_n - f) \, d\mu \right| &\le \int_X |f_n - f| \, d\mu \text{ by the Integral Comparison Test} \\ &= \int_{X_1} |f_n - f| \, d\mu + \int_{X_0 \setminus X_1} |f_n - n| \, d\mu + \int_{X \setminus X_0} |f_n - f| \, d\mu \\ &\quad \text{by Additivity over Domains, Theorem 18.12} \end{aligned}$$

The Vitali Convergence Theorem

The Vitali Convergence Theorem.
Let $(X, \mathcal{M}, \mu)$ be a measure space and let $\{f_n\}$ be a sequence of functions on X that is both uniformly integrable and tight over X. Suppose $\{f_n\} \to f$ pointwise over X. Then

$$\lim_{n\to\infty}\left(\int_X f_n\,d\mu\right) = \int_X \left(\lim_{n\to\infty} f_n\right)\,d\mu = \int_X f\,d\mu.$$

Proof. For $n \in \mathbb{N}$, $|f - f_n| \le |f| + |f_n|$ pointwise on X. If X_0 and X_1 are measurable subsets of X for which $X_1 \subset X_0$, then $X = X_1 \cup (X_0 \setminus X_1) \cup (X \setminus X_0)$. So for all $n \in \mathbb{N}$,

$$\begin{aligned}\left|\int_X (f_n - f)\,d\mu\right| &\le \int_X |f_n - f|\,d\mu \text{ by the Integral Comparison Test}\\ &= \int_{X_1} |f_n - f|\,d\mu + \int_{X_0\setminus X_1} |f_n - n|\,d\mu + \int_{X\setminus X_0} |f_n - f|\,d\mu\\ &\qquad \text{by Additivity over Domains, Theorem 18.12}\end{aligned}$$

The Vitali Convergence Theorem (continued 1)

Proof (continued).

$$\left|\int_X (f_n - f)\,d\mu\right| \leq \int_{X_1} |f_n - f|\,d\mu + \int_{X_0\setminus X_1} (|f_n| + |f|)\,d\mu + \int_{X\setminus X_0} (|f_n| + |f|)\,d\mu \text{ by monotonicity, (8) of "Lemma."} \quad (23)$$

Let $\varepsilon > 0$. Since f is integrable over X, y Proposition 18.17 there is measurable $X_0 \subset X$ (of finite measure) such that $\int_{X\setminus X_0} |f|\,d\mu < \varepsilon/6$. Since $\{f_n\}$ is tight over X, we also have for X_0 (of finite measure), $\int_{Z\setminus Z_0} |f_n|\,d\mu < \varepsilon/6$. (Technically, we have an X_0^1 for f and an X_0^2 for $\{f_n\}$, but we can define $X_0 = X_0^1 \cup X_0^2$ to get the above two claims.)

The Vitali Convergence Theorem (continued 1)

Proof (continued).

$$\left|\int_X (f_n - f)\,d\mu\right| \leq \int_{X_1} |f_n - f|\,d\mu + \int_{X_0\setminus X_1} (|f_n| + |f|)\,d\mu + \int_{X\setminus X_0} (|f_n| + |f|)\,d\mu \text{ by monotonicity, (8) of "Lemma."} \quad (23)$$

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The Vitali Convergence Theorem (continued 2)

Proof (continued). So

$$\begin{aligned}\int_{X\setminus X_0}(|f_n|+|f|)\,d\mu &= \int_{X\setminus X_0}|f_n|\,d\mu+\int_{X\setminus X_0}|f|\,d\mu \text{ by linearity, Theorem 18.11}\\ &< \frac{\varepsilon}{6}+\frac{\varepsilon}{6}=\frac{\varepsilon}{3} \text{ for } n\in\mathbb{N}. \qquad (24)\end{aligned}$$

Since $\{f_n\}$ is uniformly integrable over X, there is $\delta_1>0$ such that for measurable subset E of X: if $\mu(E)<\delta_1$ then $\int_E |f_n|\,d\mu<\varepsilon/6$ for all $n\in\mathbb{N}$. Since f is integrable over X, by Proposition 18.17 there is $\delta_2>0$ such that for measurable subset E of X: if $\mu(E)<\delta_2$ then $\int_E |f|\,d\mu<\varepsilon/6$.

The Vitali Convergence Theorem (continued 2)

Proof (continued). So

$$\begin{aligned} \int_{X\setminus X_0} (|f_n| + |f|)\, d\mu &= \int_{X\setminus X_0} |f_n|\, d\mu + \int_{X\setminus X_0} |f|\, d\mu \text{ by linearity, Theorem 18.11} \\ &< \frac{\varepsilon}{6} + \frac{\varepsilon}{6} = \frac{\varepsilon}{3} \text{ for } n \in \mathbb{N}. \qquad (24) \end{aligned}$$

Since $\{f_n\}$ is uniformly integrable over X, there is $\delta_1 > 0$ such that for measurable subset E of X: if $\mu(E) < \delta_1$ then $\int_E |f_n|\, d\mu < \varepsilon/6$ for all $n \in \mathbb{N}$. Since f is integrable over X, by Proposition 18.17 there is $\delta_2 > 0$ such that for measurable subset E of X: if $\mu(E) < \delta_2$ then $\int_E |f|\, d\mu < \varepsilon/6$. Let $\delta = \min\{\delta_1, \delta_2\}$. We now have that if $\mu(E) < \delta$, then

$$\begin{aligned} \int_E (|f_n| + |f|)\, d\mu &= \int_E |f_n|\, d\mu + \int_E |f|\, d\mu \text{ by linearity, Theorem 18.11} \\ &< \frac{\varepsilon}{6} + \frac{\varepsilon}{6} = \frac{\varepsilon}{3} \text{ for all } n \in \mathbb{N}. \qquad (25) \end{aligned}$$

The Vitali Convergence Theorem (continued 2)

Proof (continued). So

$$\begin{aligned} \int_{X\setminus X_0} (|f_n| + |f|)\, d\mu &= \int_{X\setminus X_0} |f_n|\, d\mu + \int_{X\setminus X_0} |f|\, d\mu \text{ by linearity,} \\ & \qquad\qquad\qquad\qquad \text{Theorem 18.11} \\ &< \frac{\varepsilon}{6} + \frac{\varepsilon}{6} = \frac{\varepsilon}{3} \text{ for } n \in \mathbb{N}. \qquad (24) \end{aligned}$$

Since $\{f_n\}$ is uniformly integrable over X, there is $\delta_1 > 0$ such that for measurable subset E of X: if $\mu(E) < \delta_1$ then $\int_E |f_n|\, d\mu < \varepsilon/6$ for all $n \in \mathbb{N}$. Since f is integrable over X, by Proposition 18.17 there is $\delta_2 > 0$ such that for measurable subset E of X: if $\mu(E) < \delta_2$ then $\int_E |f|\, d\mu < \varepsilon/6$. Let $\delta = \min\{\delta_1, \delta_2\}$. We now have that if $\mu(E) < \delta$, then

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The Vitali Convergence Theorem (continued 3)

Proof (continued). Since f is integrable over X, then f is finite a.e. on X by Proposition 18.9 applied to f^+ an df^-. Also, for set X_0 above we have $\mu(X_0) < \infty$. So by Egoroff's Theorem (page 364), there is a measurable subset X_1 of X_0 for which $\mu(X_0 \setminus X_1) > \delta$ and $\{f_n\}$ converges uniformly on X_1 to f (remember, Egoroff gives us that pointwise convergence is "nearly" uniform convergence). So by (25), since $\mu(X_0 \setminus X_1) < \delta$,

$$\int_{X_0 \setminus X_1} (|f_n| + |f|)\, d\mu < \frac{\varepsilon}{3} \text{ for } n \in \mathbb{N}. \tag{26}$$

The Vitali Convergence Theorem (continued 3)

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Since $\{f_n\}$ converges uniformly to f on X_1, a set of finite measure (since X_0 is finite measure), there is $N \in \mathbb{N}$ for which

$$\begin{aligned} \int_{X_1} |f_n - f|\, d\mu &\leq \sup_{x \in X_1} \{|f_x(x) - |f(x)|\} \mu(X_1) \text{ by Integral Comp. Test} \\ &< \frac{\varepsilon}{3} \text{ for all } n \geq N. \end{aligned} \tag{27}$$

The Vitali Convergence Theorem (continued 3)

Proof (continued). Since f is integrable over X, then f is finite a.e. on X by Proposition 18.9 applied to f^+ an df^-. Also, for set X_0 above we have $\mu(X_0) < \infty$. So by Egoroff's Theorem (page 364), there is a measurable subset X_1 of X_0 for which $\mu(X_0 \setminus X_1) > \delta$ and $\{f_n\}$ converges uniformly on X_1 to f (remember, Egoroff gives us that pointwise convergence is "nearly" uniform convergence). So by (25), since $\mu(X_0 \setminus X_1) < \delta$,

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Since $\{f_n\}$ converges uniformly to f on X_1, a set of finite measure (since X_0 is finite measure), there is $N \in \mathbb{N}$ for which

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The Vitali Convergence Theorem (continued 4)

Proof (continued). Applying inequalities (24), (26), and (27) to inequality (23) gives $\left|\int_X (f_n - f)d\mu\right| < \varepsilon$ for all $n \geq N$. That is, by linearity (Theorem 18.12)

$$\left|\int_X f_n\,d\mu - \int_X f\,d\mu\right| < \varepsilon \text{ for all } n \geq N,$$

or

$$\lim_{n\to\infty} \int_X f_n\,d\mu = \int_X f\,d\mu.$$

□