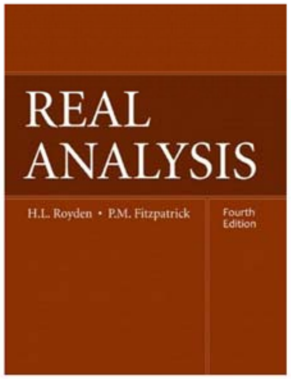
Real Analysis

Chapter 18. Integration Over General Measure Spaces

18.4. The Radon-Nikodym Theorem—Proofs of Theorems



Proposition 18.19

Proposition 18.19. Let $(X, \mathcal{M}, \mu)$ be a measure space and ν a finite measure on the measurable space $(X, \mathcal{M})$. Then ν is absolutely continuous with respect to μ if and only if for each $\epsilon > 0$ there is a $\delta > 0$ such that for any set $E \in \mathcal{M}$, if $\mu(E) < \delta$ then $\nu(E) < \epsilon$.

Proof. Suppose the ε/δ condition holds and $\mu(E) = 0$. Then for all $\varepsilon > 0$, we have $\nu(E) < \varepsilon$ and hence $\nu(E) = 0$. So ν is absolutely continuous with respect to μ.

Next, suppose ν is absolutely continuous with respect to μ and the ε/δ condition *does not* hold. Then there is $\varepsilon_0 > 0$ and a sequence of sets $\{E_n\} \subset \mathcal{M}$ such that for each $n \in \mathbb{N}$, $\mu(E_n) < 1/2^n$ while $\nu(E_n) \geq \varepsilon_0$ (otherwise, we could eventually rake $\delta_0 = 1/2^n$ for some $n \in \mathbb{N}$ and the ε/δ condition *would* hold). For each $n \in \mathbb{N}$, define $A_n = \cup_{k=n}^{\infty} E_k$. Then $\{A_n\}$ is a descending sequence of sets in $\mathcal{M}$.

Proposition 18.19 (continued 1)

Proof (continued). By the monotonicity of ν (Proposition 17.1) and the countable subadditivity of μ (any measure is countably additive by definition, and so is also countably subadditive on the σ-algebra of measurable sets): $\nu(A_n) = \nu(\cup_{k=n}^{\infty} E_k) \geq E_n) \geq \varepsilon_0$ and

$$\mu(A_n) = \mu(\cup_{k=n}^{\infty} E_k) \leq \sum_{k=n}^{\infty} \mu(E_k) < \sum_{k=n}^{\infty} \frac{1}{2^k} = \frac{1}{2^{n-1}}$$

for all $n \in \mathbb{N}$. Define $A_\infty = \cup_{k=1}^{\infty} A_k$. Then

$$\begin{aligned} \mu(A_\infty) &= \mu(\cap_{k=1}^{\infty} A_k) \leq (A_n) \text{ by monotonicity} \\ &< \frac{1}{2^{n-1}} \text{ for all } n \in \mathbb{N}, \end{aligned}$$

and so $\mu(A_\infty) = 0$. Next, $\nu(X) < \infty$ since ν is a finite measure by hypothesis and so $\nu(A_1) \leq \nu(X) < \infty$ by monotonicity.

Proposition 18.19 (continued 2)

Proposition 18.19. Let $(X, \mathcal{M}, \mu)$ be a measure space and ν a finite measure on the measurable space $(X, \mathcal{M})$. Then ν is absolutely continuous with respect to μ if and only if for each $\epsilon > 0$ there is a $\delta > 0$ such that for any set $E \in \mathcal{M}$, if $\mu(E) < \delta$ then $\nu(E) < \epsilon$.

Proof (continued). Then, by the Continuity of Measure of ν (Proposition 17.2) and the fact that $\nu(A_n) \geq \varepsilon_0 > 0$ for all $n \in \mathbb{N}$, we have

$$\nu(A_\infty) = \nu\left(\lim_{n\to\infty} A_n\right) = \lim_{n\to\infty} \nu(A_n) \geq \varepsilon_0 > 0.$$

But we have hypothesized the absolute continuity of ν with respect to μ, and here we have $\mu(A_\infty) = 0$ but $\nu(A_\infty) > 0$, so this is a contradiction (to our assumption that the ε/δ condition does not hold). Therefore, the ε/δ condition does hold. □

The Radon-Nikodym Theorem

The Radon-Nikodym Theorem. Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space and ν a σ-finite measure defined on the measurable space $(X, \mathcal{M})$ that is absolutely continuous with respect to μ. Then there is a nonnegative f on X that is measurable with respect to $\mathcal{M}$ for which

$$\nu(E) = \int_E f \, d\mu \text{ for all } E \in \mathcal{M}.$$

The function f is unique in the sense that if g is any nonnegative measurable function on X for which $\nu(E) = \int_E g \, d\mu$ for all $E \in \mathcal{M}$, then $g = f$ μ-a.e.

Proof. We consider the case where both μ and ν are finite measures and cover th eσ-finite case in Exercise 18.49. If $\nu(E) = 0$ for all $E \in \mathcal{M}$ then the claim holds with $f = 0$ on X. So without loss of generality we can assume ν does not vanish on all of $\mathcal{M}$.

The Radon-Nikodym Theorem (continued 1)

Proof (continued). We first prove that there is nonnegative measurable f on X for which

$$\int_X f \, d\mu > 0 \text{ and } \int_E f \, d\mu \le \nu(E) \text{ for all } E \in \mathcal{M}. \quad (32)$$

For $\lambda > 0$, consider the finite signed measure $\nu - \lambda\mu$ (since μ and ν are finite, $\nu - \lambda\mu$ satisfies the definition of signed measure; see Section 17.21). By the Hahn Decomposition Theorem, there is a Hahn decomposition $\{P_\lambda, N_\lambda\}$ for $\nu - \lambda\mu$ where $X = P_\lambda \cup N_\lambda$, $P_\lambda \cap N_\lambda = \varnothing$, P_λ is a positive $\nu = \lambda\mu$ measure set, and N_λ is a negative $\nu - \lambda\mu$ measure set.

ASSUME $\mu(P_\lambda) = 0$ for all $\lambda > 0$. Then for any measurable $E \subset P_\lambda$ we have absolute continuity, $\nu(E) = 0$. Since N_λ is a negative set for $\nu - \lambda\mu$ then for *any* $E \in \mathcal{M}$,

$$(\nu - \lambda\mu)(E) = (\nu - \lambda\mu)((E \cap P_\lambda) \cup (E \cap N_\lambda)) = \nu((E \cap P_\lambda) \cup (E \cap N_\lambda))$$
$$-\lambda\mu((E \cap P_\lambda) \cup (E \cap N_\lambda)) = \nu((E \cap P_\lambda) + \nu(E \cap N_\lambda)$$
$$-\lambda\mu(E \cap P_\lambda) - \lambda\mu(E \cap N_\lambda) = \nu(E \cap N_\lambda) - \lambda\mu(E \cap N_\lambda) \le 0 \ldots$$

The Radon-Nikodym Theorem (continued 2)

Proof (continued). ...since by the definition of "negative set" we have every subset of N_λ has nonpositive measure, and this holds for all $\lambda > 0$. That is, $\nu(E) \le \lambda\mu(E)$ for all $E \in \mathcal{M}$ and for all $\lambda > 0$. Since $\lambda > 0$ is arbitrary and $\mu(E) \le \mu(X) < \infty$ we must have $\nu(E) = 0$ for all $E \in \mathcal{M}$. But this is a CONTRADICTION to the fact that ν does not vanish on all o f$\mathcal{M}$. So the assumption that $\mu(P_\lambda) = 0$ for all $\lambda > 0$ is false and there is some $\lambda_0 > 0$ such that $\mu(P_{\lambda_0}) > 0$.

Define $f = \lambda_0 \chi_{P_{\lambda_0}}$. Then $\int_X f \, d\mu = \int_X \lambda_0 \chi_{P_{\lambda_0}} \, d\mu = \lambda_0 \mu(P_{\lambda_0}) > 0$ and since $\nu - \lambda_0\mu$ is positive on P_{λ_0} then

$$\begin{aligned} \int_E f \, d\mu &= \int_E \lambda_0 \chi_{P_{\lambda_0}} \, d\mu = \lambda_0 \mu(P_{\lambda_0} \cap E) \\ &\le \nu(P_{\lambda_0} \cap E) \text{ since } P_{\lambda_0} \cap E \subset P_{\lambda_0} \text{ and so} \\ &\quad \nu(P_{\lambda_0} \cap E) = \lambda_0 \mu(P_{\lambda_0} \cap E) \ge 0 \\ &\le \nu(E) \text{ by monotonicity.} \end{aligned}$$

The Radon-Nikodym Theorem (continued 3)

Proof (continued). Therefore (32) holds for $f = \lambda_0 \chi_{P_{\lambda_0}}$. Define $\mathcal{F}$ to be the collection of nonnegative measurable functions on X for which $\int_E f \, d\mu \le \nu(E)$ for all $E \in \mathcal{M}$ (so $\mathcal{F}$ is nonempty since $\lambda_0 \chi_{P_{\lambda_0}} \in \mathcal{F}$) and then define $M = \sup_{f \in \mathcal{F}} \int_X f \, d\mu$. Notice that $M > 0$ since $\lambda_0 \chi_{P_{\lambda_0}} \in \mathcal{F}$. We now show that there is $f \in \mathcal{F}$ for which $\int_X f \, d\mu = M$ and that $\nu(E) = \int_E f \, d\mu$ for all $E \in \mathcal{M}$ for any such f.

If $g, h \in \mathcal{F}$ then with $E_1 = \{x \in E \mid g)x) < h(x)\}$ and $E_2 = \{x \in E \mid g(x) \ge h(x)\}$ we have

$$\begin{aligned} \int_E \max\{g, h\} \, d\mu &= \int_{E_1} h \, d\mu + \int_{E_2} g \, g\nu \\ &\le \nu(E_1) + \nu(E_2) \text{ by the definition of } \mathcal{F} \\ &= \nu(E), \end{aligned}$$

so that $\max\{g, h\} \in \mathcal{F}$.

The Radon-Nikodym Theorem (continued 4)

Proof (continued). Next, select a sequence $\{f_n\} \in \mathcal{F}$ for which $\lim_{n\to\infty} (\int_X f_n \, f\mu) = M$ (such a sequence exists by the definition of supremum). We may assume $\{f_n\}$ is a pointwise increasing sequence of functions (or else we can replace f_n by $\max\{f_1, f_2, \ldots, f_n\}$, since we now know $\max\{f_1, f_2, \ldots, f_n\} \in \mathcal{F}$, to get an increasing sequence). Define $f(x) = \lim_{n\to\infty} f_n(x)$ for each $x \in X$. Since $\mathcal{F}$ consists only of nonnegative functions (by the definition of $\mathcal{F}$) and $\{f_n\}$ is monotone, then by the Monotone Convergence Theorem (of Section 18.2),

$$\int_X f \, d\mu = \int_X \lim_{n\to\infty} f_n \, d\mu = \lim_{n\to\infty} \left(\int_X f_n \, d\mu \right) = M.$$

Also $\nu(E) \geq \int_E f_n \, d\mu$ for all $E \in \mathcal{M}$ (by the definition of $\mathcal{F}$) and so by the Monotone Convergence Theorem

$$\nu(E) \geq \lim_{n\to\infty} \left(\int_E f_n \, d\mu \right) = \int_E \left(\lim_{n\to\infty} f_n \right) d\mu = \int_E f \, d\mu$$

for all $E \in \mathcal{M}$, and so $f \in \mathcal{F}$, as desired.

The Radon-Nikodym Theorem (continued 5)

Proof (continued). Define $\eta(E) = \nu(E) - \int_E f \, d\mu$ for all $E \in \mathcal{M}$. We have assumed ν is a finite measure and so $\nu(X) < \infty$. Therefore $\int_X f \, d\mu \leq \nu(X) < \infty$. As shown above, $\nu(E) \geq \int_E f \, d\mu$ for all $E \in \mathcal{M}$, so $\eta(E) \geq 0$ for all $E \in \mathcal{M}$. Now ν is countably additive since it is a measure and so by Theorem 18.13, "Countable Additivity Over Domains of Integration," η is countably additive and so η is a measure on $\mathcal{M}$. Also, for $E \in \mathcal{M}$ with $\mu(E) = 0$ we have $\nu(E) = 0$ since ν is absolutely continuous with respect to μ and $\int_E f \, d\mu = 0$ so that $\eta(E) = 0$; that is, η is absolutely continuous with respect to μ.

ASSUME there is some set $E \in \mathcal{M}$ for which $\eta(E) > 0$. Then, as argued above for ν, we can find $\hat{f}$ a nonnegative function such that $\int_X \hat{f} \, d\mu > 0$ and $\int_E \hat{f} \, d\mu \leq \eta(E)$ for all $E \in \mathcal{M}$ (we had the function $\lambda_0 \chi_{P_{\lambda_0}}$ as such a function in $\mathcal{F}$ for ν). So from the definition of η we have for this $\hat{f}$ that $\int_E \hat{f} \, d\mu \leq \eta(E) = \nu(E) - \int_E f \, d\mu$ for all $E \in \mathcal{M}$ (where f is the function $f(x) = \lim_{n\to\infty} f_n(x)$ defined above).

The Radon-Nikodym Theorem (continued 6)

Proof (continued). Then $\int_X (f + \hat{f}) \, d\mu = \int_X f \, d\mu + \int_E \hat{f} \, d\mu > 0$ and for all $E \in \mathcal{M}$, so $\int_E (f + \hat{f}) \, d\mu \leq \nu(E) - \int_E f \, d\mu \leq \nu(E)$, so that $f + \hat{f} \in \mathcal{F}$. But then $\int_X (f + \hat{f}) \, d\mu > \int_X f \, d\mu = M$, a CONTRADICTION to the definition o fM. So the assumption that $\eta(E) > 0$ for some $E \in \mathcal{M}$ is false and hence $\eta(E) = 0$ for all $E \in \mathcal{M}$. Therefore, $\nu(E) = \int_E f \, d\mu$ for all $E \in \mathcal{M}$, as claimed.

For uniqueness, suppose f_1 and f_2 both satisfy $\nu(E) = \int_E f_1 \, d\mu = \int_E f_2 \, d\mu$ for all $E \in \mathcal{M}$. Since ν is a finite measure then f_1 and f_2 are both integrable. Also,

$$\int_E (f_1 - f_2) \, d\mu = \int_E f_1 \, d\mu - \int_E f_2 \, d\mu = 0$$

and so by Exercise 18.32, $f_1 = f_2$ μ-a.e., as claimed. □

The Lebesgue Decomposition Theorem

The Lebesgue Decomposition Theorem.
Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space and ν a σ-finite measure on the measurable space $(X, \mathcal{M})$. Then there is a measure ν_0 on $\mathcal{M}$ which is singular with respect to μ, and a measure ν_1 on $\mathcal{M}$ which is absolutely continuous with respect to μ, for which $\nu = \nu_0 + \nu_1$. The measures ν_0 and ν_1 are unique.

Proof. Define $\lambda = \mu + \nu$. In Exercise 18.58 it is to be shown that if g is nonnegative and measurable with respect to $\mathcal{M}$, then

$$\int_E g \, d\lambda = \int_e g \, d\mu + \int_E g \, d\nu \text{ for all } E \in \mathcal{M}.$$

Since μ and ν are σ-finite (that is, X is a union of a countable number of measurable sets of finite measure) then $\lambda = \mu + \nu$ is also σ-finite. Moreover, if $\lambda(E) = 0$ then $\mu(E) + \nu(E) = 0$ and so $\mu(E) = 0$; that is, μ is absolutely continuous with respect to λ.

The Lebesgue Decomposition Theorem (continued 1)

Proof (continued). So by the Radon-Nikodym Theorem there is a nonnegative measurable functions f for which

$$\mu(E) = \int_E f\,d\lambda = \int_E f\,d(\mu+\nu) = \int_E f\,d\mu + \int_E f\,d\nu \text{ for all } E \in \mathcal{M}. \quad (37)$$

Define $X_+ = \{x \in X \mid f(x) > 0\}$ and $X_0 = \{x \in X \mid f(x) = 0\}$. Since f is a measurable function then X_+ and X_0 are measurable. Define $\nu_0(E) = \nu(E \cap X_0)$ and $\nu_1(E) = \nu(E \cap X_+)$ for all $E \in \mathcal{M}$. Then $\nu = \nu_0 + \nu_1$ on $\mathcal{M}$, as claimed. Now $\mu(X_0) = \int_{X_0} f\,d\lambda = \int_{X_0} 0\,d\lambda = 0$ and $\nu_)(X_+) = \nu(X_+ \cap X_0) = \nu(\varnothing) = 0$. So (by definition) μ and ν_0 are mutually singular (that is, $\mu \perp \nu_0$), as claimed.

The Lebesgue Decomposition Theorem (continued 2)

The Lebesgue Decomposition Theorem.
Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space and ν a σ-finite measure on the measurable space $(X, \mathcal{M})$. Then there is a measure ν_0 on $\mathcal{M}$ which is singular with respect to μ, and a measure ν_1 on $\mathcal{M}$ which is absolutely continuous with respect to μ, for which $\nu = \nu_0 + \nu_1$. The measures ν_0 and ν_1 are unique.

Proof (continued). Next, we show ν_1 is absolutely continuous with respect to μ. Let $\mu(E) = 0$. Then $\int_E f\,d\mu = 0$. Therefore by (37) $\int_E f\,d\nu = 0$ and so (by additivity) $0 = \int_E f\,d\nu = \int_{E\cap X_0} f\,d\nu + \int_{E\cap X_+} f\,f\nu$. Since $f = 0$ on $E \cap X_0$ then $\int_{E\cap X_0} f\,d\nu = 0$ and so $\int_{E\cap X_+} f\,d\nu = 0$. But $f > 0$ on $E \cap X_+$ and so by Exercise 18.19 $f = 0$ ν-a.e. on $E \cap X_+$. So we must have $\nu(E \cap X_+) = 0$. That is, $\nu_1(E) = 0$. So ν_1 is absolutely continuous with respect to μ, as claimed. Uniqueness is to be shown in Exercise 18.55. □