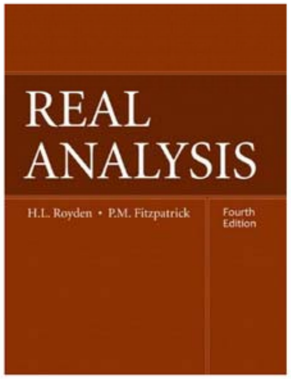
Real Analysis

Chapter 18. Integration Over General Measure Spaces
18.5. The Nikodym Metric Space: The Vitali-Hahn-Saks Theorem—Proofs of Theorems



Lemma 18.5.A

Lemma 18.5.A. The map $\rho_\mu : \mathcal{M}/_\simeq \times \mathcal{M}/_\simeq \to \mathbb{R}$ is well-defined and is a metric on $\mathcal{M}/_\simeq$.

Proof. First, we show ρ_μ is well-defined. Let $A_1 \simeq A_2$ and $B_1 \simeq B_2$; that is, $\mu(A_1 \triangle A_2) = \mu(B_1 \triangle B_2) = 0$. So

$$\begin{aligned} \rho_\mu([A_1],[B_2]) &= \mu(A_1 \triangle B_1) \\ &= \mu((A_1 \triangle A_2) \triangle (A_2 \triangle B_1)) \text{ since } (A \triangle B) \triangle (B \triangle C) = A \triangle C \\ &= \nu((A_1 \triangle A_2) \triangle ((A_2 \triangle B_2) \triangle (B_2 \triangle B_1))) \\ &\quad \text{ since } (A \triangle B) \triangle (B \triangle C) = A \triangle C. \end{aligned}$$

Now $\mu(B_1 \triangle B_2) = \mu(B_2 \triangle B_1) = 0$, so $(A_\triangle B_2) \triangle (B_2 \triangle B_1)$ consists of almost all elements of $A_1 \triangle B_2$ along with some subset of $B_2 \triangle B_1$ (which must have μ-measure 0), so that $\mu((A_2 \triangle B_2) \triangle (B_2 \triangle B_1)) = \mu(A_2 \triangle B_2)$.

Lemma 18.5.A (continued 1)

Proof (continued). Similarly, since $\mu(A_1 \triangle A_2) = 0$,

$$\begin{aligned} &\mu((A_1 \triangle A_2) \triangle ((A_2 \triangle B_2) \triangle (B_2 \triangle B_1)) \\ &= \mu((A_2 \triangle B_2) \triangle (B_2 \triangle B_1)) = \mu(A_2 \triangle B_2). \end{aligned}$$

Hence, $\rho_\mu([A_1],[B_1]) = \mu(A_2 \triangle B_2) = \rho_\mu([A_2],[B_2])$ and ρ_μ is well-defined.

We now check the definition of "metric" (see Section 9.1). Let $[A],[B],[C] \in \mathcal{M}/_\simeq$.

(i) $\rho_\mu([A],[B]) = \mu(A \triangle B) \geq 0$,
(ii) $\rho_\mu([A],[B]) = 0$ if and only if $\mu(A \triangle B) = 0$ if and only if $A \simeq B$ if and only if $[A] = [B]$,
(iii) $\rho_\mu([A],[B]) = \mu(A \triangle B) = \mu(B \triangle A) = \rho_\mu([B],[A])$,
(iv) $\rho_\mu([A],[B]) + \rho_\mu([B],[C])$

$$= \mu(A \triangle B) + \mu(B \triangle C) \geq \mu((A \triangle B) \cup (B \triangle C))$$

by subadditivity

Lemma 18.5.A (continued 2)

Proof (continued).

$$\begin{aligned} &\geq \mu((A \triangle B) \triangle (B \triangle C)) \text{ by monotonicity since} \\ &\quad (A \triangle B) \triangle (B \triangle C) = ((A \triangle B) \setminus (B \triangle C)) \cup ((B \triangle C) \setminus (A \triangle B)) \\ &\quad \subset (A \triangle B) \cup (B \triangle C) \\ &= \mu(A \triangle C) \text{ since } (A \triangle B) \triangle (B \triangle C) = A \triangle C \\ &= \rho_\mu([A],[C]), \end{aligned}$$

and the Triangle Inequality holds.
Therefore ρ_μ is a metric. □

Theorem 18.21

Theorem 18.21. Let $(X, \mathcal{M}, \mu)$ be a finite measure space. Then the Nikodym metric space $(\mathcal{M}, \rho_\mu)$ is complete; that is, every Cauchy sequence converges.

Proof. For $A, B \in \mathcal{M}$, we have $\chi_{A \triangle B} = |\chi_A - \chi_B|$, so

$$\mu(A \triangle B) = \int_X |\chi_A - \chi_B| \, d\mu. \quad (41)$$

Define $T : \mathcal{M} \to L^1(X, \mu)$ by $T(E) = \chi_E$ (we need the fact that μ is a finite measure here). Then T is an isometry between the metric spaces since by (41),

$$\rho_\mu(A, B) = \mu(A \triangle B) = \int_X |\chi_A - \chi_B| \, d\mu = \|T(A) - T(B)\|_1 \text{ for all } A, B \in \mathcal{M}.$$

Let $\{A_n\}$ be a Cauchy sequence in (M, ρ_μ). Then $\{T(A_n)\}$ is a Cauchy sequence in $L^1(X, \mu)$ since T is an isometry.

Theorem 18.21 (continued)

Theorem 18.21. Let $(X, \mathcal{M}, \mu)$ be a finite measure space. Then the Nikodym metric space $(\mathcal{M}, \rho_\mu)$ is complete; that is, every Cauchy sequence converges.

Proof (continued). By the Riesz-Fischer Theorem (we actually need the version for measure spaces which is not stated until Section 19.1) there is $f \in L^1(X, \mu)$ such that $\{T(A_n)\} \to f$ in $L^1(X, \mu)$, and there is a subsequence of $\{T(A_n)\}$ that converges pointwise to f μ-a.e. on X. Since each $T(A_n)$ is a characteristic function and so only takes on the values 0 and 1, if we define A_0 to be the set of points in X at which the pointwise convergent subsequence converges to 1, then the limit function f satisfies $f = \chi_{A_0}$ μ-a.e. on X. Since $\{T(A_n)\}$ converges to $T(A_0) = \chi_{A_0}$ in $L^1(X, \mu)$ then (because T is an isometry) $\{A_n\}$ converges to A_0 in $(\mathcal{M}, \rho_\mu)$. Since $\{A_n\}$ is an arbitrary Cauchy sequence in $(\mathcal{M}, \rho_\mu)$, then $(\mathcal{M}, \rho_\mu)$ is complete. □

Lemma 18.22

Lemma 18.22. Let $(X, \mathcal{M}, \mu)$ be a finite measure space and ν a finite measure on $\mathcal{M}$. Let E_0 be a measurable set and $\varepsilon > 0$ and $\delta > 0$ be such that for any measurable set E,

$$\text{if } \rho_\mu(E, E_0) < \delta \text{ then } |\nu(E) - \nu(E_0)| < \varepsilon/4.$$

Then for any measurable sets A and B,

$$\text{if } \rho_\mu(A, B) < \delta \text{ then } |\nu(A) - \nu(B)| < \varepsilon.$$

Proof. We first show

$$\text{if } \rho_\mu(A, B) < \delta \text{ then } |\nu(A) - \nu(B)| < \varepsilon/2. \quad (45)$$

Now if $D \subset C$ then $C \triangle D = (C \setminus D) \cup (D \setminus C) = C \setminus D$. Let $A \in \mathcal{M}$ and $\rho(A, \varnothing) = \mu(A \triangle \varnothing) = \mu(A) < \delta$. We have

$$(E_0 \setminus A) \triangle E_0 = (E_0 \setminus (E_0 \setminus A)) \cup (E_0 \setminus (E_0 \setminus A)) = E_0 \setminus (E_0 \setminus A) = E_0 \cap A \subset A.$$

Hence $\rho_\mu(E_0 \setminus A, E_0) = \mu((E_0 \setminus A) \triangle E_0) = \mu(A) < \delta$, and so by hypothesis $|\nu(E_0) - \nu(E_0 \setminus A)| = \nu(E_0) - \nu(E_0 \setminus A) < \varepsilon/4$.

Lemma 18.22 (continued 1)

Proof (continued). By the excision property (Proposition 17.1), since ν is a finite measure,

$$\nu(A \cap E_0) = \nu(E_0 \setminus (E_0 \setminus A)) = \nu(E_0) - \nu(E_0 \setminus A) < \varepsilon/4.$$

Now observe that

$$E_0 \triangle (E_0 \cup (A \setminus E_0)) = (E_0 \setminus (E_0 \cup (A \setminus E_0))) \cup ((E_0 \cup (A \setminus E_0)) \setminus E_0)$$

$$= \varnothing \cup ((E_0 \cup (A \setminus E_0)) \setminus E_0) = (E_0 \cup (A \setminus E_0)) \setminus E_0 = A \setminus E_0 \subset A.$$

So

$$\rho_\mu(E_0, E_0 \cup (A \setminus E_0)) = \mu(E_0 \triangle (E_0 \cup (A \setminus E_0))) = \mu(A \setminus E_0) \le \mu(A) < \delta.$$

Thus, again by the excision property, and hypothesis

$$\nu(A \setminus E_0) = \nu((E_0 \cup (A \setminus E_0)) \setminus E_0) = |\nu(E_0 \cup (A \setminus E_0)) - \nu(E_0)|$$

$$= \nu(E_0 \cup (A \setminus E_0)) - \nu(E_0) < \varepsilon/4.$$

Lemma 18.22 (continued 2)

Proof (continued). Therefore

$$\begin{aligned} \nu(A) &= \nu((A\cap E_0)\cup(A\setminus E_0)) \\ &= \nu(A\cap E_0)+\nu(A\setminus E_0) \text{ by additivity} \\ &< \varepsilon/4+\varepsilon/4=\varepsilon/2. \end{aligned}$$

So (45) holds.

Now let $A, B \in \mathcal{M}$. Then

$$\begin{aligned} \nu(A)-\nu(B) &= \nu((A\cap B)\uplus(A\setminus B))-\nu((A\cap B)\uplus(B\setminus A)) \\ &= \nu(A\cap B)+\nu(A\setminus B)=\nu(A\cap B)-\nu(B\setminus A) \text{ by additivity} \\ &= \nu(A\setminus B)-\nu(B\setminus A), \end{aligned}$$

so $|\nu(A)-\nu(B)|=|\nu(A\setminus B)-\nu(B\setminus)|\le|\nu(A\setminus B)|+|\nu(B\setminus A)|$.

Lemma 18.22 (continued 3)

Lemma 18.22. Let $(X,\mathcal{M},\mu)$ be a finite measure space and ν a finite measure on $\mathcal{M}$. Let E_0 be a measurable set and $\varepsilon>0$ and $\delta>0$ be such that for any measurable set E,

$$\text{if } \rho_\mu(E,E_0)<\delta \text{ then } |\nu(E)-\nu(E_0)|<\varepsilon/4.$$

Then for any measurable sets A and B,

$$\text{if } \rho_\mu(A,B)<\delta \text{ then } |\nu(A)-\nu(B)|<\varepsilon.$$

Proof (continued). Suppose $\rho_\mu(A,B)=\mu(A\triangle B)<\delta$. Since $A\setminus B, B\setminus A\subset A\triangle B$ then $\rho_\mu(A\setminus B,\varnothing)=\mu(A\setminus B)\le\mu(A\triangle B)<\delta$ and so $\rho_\mu(B\setminus A,\varnothing)=\mu(B\setminus A)\le\nu(A\triangle B)<\delta$ by monotonicity (Proposition 17.1). So by (45), we have $\nu(A\setminus B)<\varepsilon/2$ and $\nu(B\setminus A)<\varepsilon/2$. Hence, $|\nu(A)-\nu(B)|\le|\nu(A\setminus B)|+|\nu(B\setminus A)|<\varepsilon$, as claimed. □

Proposition 18.23

Proposition 18.23. Let $(X,\mathcal{M},\mu)$ be a finite measure space and ν a finite measure on $\mathcal{M}$ that is absolutely continuous with respect to μ. Then ν induces a properly defined (i.e., "well-defined"), uniformly continuous function on the Nikodym metric space associated with $(X,\mathcal{M},\mu)$.

Proof. Since finite measure space ν is absolutely continuous with respect to μ then, by Proposition 18.19, for every $\varepsilon>0$ there is $\delta>0$ such that for any $E\in\mathcal{M}$, if $\mu(E)<\delta$ then $\nu(E)<\varepsilon$. Since $\mu(E)=\rho_\mu(E,\varnothing)$, this means that $\nu:\mathcal{M}\to\mathbb{R}$ is continuous at $E_0=\varnothing$ (where ν maps metric space $(\mathcal{M},\rho_\mu)$ to metric space $(\mathbb{R},|\cdot|)$. By Lemma 18.22, ν is uniformly continuous on $(\mathcal{M},\rho_\mu)$. □

Theorem 18.25

Theorem 18.25. Let $(X,\mathcal{M},\mu)$ be a finite measure space and $\{\nu_n\}$ a sequence of finite measures on $\mathcal{M}$ that is uniformly absolutely continuous with respect to μ. If $\{\nu_n\}$ converges setwise on $\mathcal{M}$ to ν, then ν is a measure of $\mathcal{M}$ that is absolutely continuous with respect to μ.

Proof. Since each ν_n is nonnegative then ν is nonnegative. Let $\{E_k\}_{k=1}^m$ be disjoint. Then

$$\nu\left(\uplus_{k=1}^m E_k\right)=\lim_{n\to\infty}\nu_n\left(\uplus_{k=1}^m E_k\right)=\lim_{n\to\infty}(\nu_n(E_1)+\nu_n(E_2)+\cdots\nu_n(E_m))$$

$$=\lim_{n\to\infty}\nu_n(E_1)+\lim_{n\to\infty}\nu_n(E_2)+\cdots+\lim_{n\to\infty}\nu_n(E_m)=\nu(E_1)+\nu(E_2)+\cdots+\nu(E_m)$$

and so ν is finite additive. If $A\subset B$ then $\nu_n(A)\le\nu_n(B)$, so $\nu(A)=\lim_{n\to\infty}\nu_n(A)\le\lim_{n\to\infty}\nu_n(B)=\nu(B)$ and so ν is monotone. For countable additivity, consider disjoint $\{E_k\}_{k=1}^\infty$. If $\nu(E_k)=\infty$ for some E_k, then $\nu\left(\uplus_{k=1}^\infty E_k\right)\ge\nu(E_k)=\infty=\sum_{k=1}^\infty\nu(E_k)$ and so countable additivity holds. So we can assume without loss of generality that $\nu(E_k)<\infty$ for all $k\in\mathbb{N}$.

Theorem 18.25 (continued 1)

Proof (continued). By finite additivity, for each $n \in \mathbb{N}$

$$\nu\left(\cup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{n} \nu(E_k) + \nu\left(\cup_{k=n+1}^{\infty} E_k\right). \quad (47)$$

Let $\varepsilon > 0$. Since $\{\nu_n\}$ is uniformly absolutely continuous with respect fo μ, there is $\delta > 0$ such that for $E \in \mathcal{M}$ and for all $n \in \mathbb{N}$,

$$\text{if } \mu(E) < \delta \text{ then } \nu_n(E) < \varepsilon/2. \quad (48)$$

Since $\nu(E) = \lim_{n\to\infty} \nu_n(E)$ for all $E \in \mathcal{M}$, then $\nu_n(E) < \varepsilon/2$ implies $\nu(E) \le \varepsilon/2 < \varepsilon$. Since $\mu(X) < \infty$ and μ is countably additive then $\mu\left(\cup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mu(E_k) < \infty$ and so there is $N \in \mathbb{N}$ for which $\mu\left(\cup_{k+N+1}^{\infty} E_k\right) = \sum_{k=N+1}^{\infty} \mu(E_k) < \delta$ (since the "tail" of a convergent series must be small) and so $\nu\left(\cup_{k=N+1}^{\infty} E_n\right) < \varepsilon$. So from (47),

$$0 \le \nu\left(\cup_{k=1}^{\infty} E_k\right) - \sum_{k=1}^{N} \nu(E_k) = \nu\left(\cup_{k=N+1}^{\infty} E_k\right) < \varepsilon.$$

Theorem 18.25 (continued 2)

Theorem 18.25. Let $(X, \mathcal{M}, \mu)$ be a finite measure space and $\{\nu_n\}$ a sequence of finite measures on $\mathcal{M}$ that is uniformly absolutely continuous with respect to μ. If $\{\nu_n\}$ converges setwise on $\mathcal{M}$ to ν, then ν is a measure of $\mathcal{M}$ that is absolutely continuous with respect to μ.

Proof (continued). Therefore

$$\lim_{N\to\infty} \left(\sum_{k=1}^{N} \nu(E_k)\right) = \sum_{k=1}^{\infty} \nu(E_k) = \nu\left(\cup_{k=1}^{\infty} E_k\right),$$

as claimed. So ν is a measure on $\mathcal{M}$.

From (48) (based on the uniformly absolute continuity of $\{\nu_n\}$ with respect to μ) we see that if $\mu(E) = 0$ then $\nu_n(E) = 0$ for all $n \in \mathbb{N}$ and so $\nu(E) = \lim_{n\to\infty} \nu_n(E) = 0$. So ν is absolutely continuous with respect to μ. □

The Vitali-Hahn-Saks Theorem

The Vitali-Hahn-Saks Theorem.
Let $(X, \mathcal{M}, \mu)$ be a finite measure space and $\{\nu_n\}$ a sequence of finite measures on $\mathcal{M}$, each if which is absolutely continuous with respect to μ. Suppose that $\{\nu_n(X)\}$ is bounded and $\{\nu_n\}$ converges setwise on $\mathcal{M}$ to ν. Then the sequence $\{\nu_n\}$ is uniformly continuous with respect to μ. Moreover, ν is a finite measure on $\mathcal{M}$ that is absolutely continuous with respect to μ.

Proof. By Proposition 18.23, $\{\nu_n\}$ induces a sequence of (uniformly) continuous functions on the Nikodym metric space where $\nu_n : \mathcal{M} \to \mathbb{R}$ (and the domain of ν_n is the metric space $(\mathcal{M}, \rho_\mu)$). Since $\{\nu_n(X)\}$ is bounded for ν is finite on $\mathcal{M}$. By Theorem 18.21, $(\mathcal{M}, \rho_\mu)$ is a complete metric space.

The Vitali-Hahn-Saks Theorem (continued)

Proof (continued). So by Theorem 10.7 (a consequence of the Baire Category Theorem), there is a set $E_0 \in \mathcal{M}$ for which the sequence $\{\nu_n : \mathcal{M} \to \mathbb{R}\}$ is equicontinuous at E_0; that is, for each $\varepsilon > 0$ there is $\delta > 0$ such that for each measurable set E and $n \in \mathbb{N}$,

$$\text{if } \rho_\mu(E, E_0) < \delta \text{ then } |\nu_n(E) - \nu_n(E_0)| < \varepsilon.$$

Since each ν_n is finite, Lemma 18.22 implies that for each $\varepsilon > 0$ there is $\delta > 0$ such that for each $E \in \mathcal{M}$ and each $n \in \mathbb{N}$

$$\text{if } \rho_\mu(E, \varnothing) = \mu(E) < \delta \text{ then } |\nu_n(E) - \nu_n(\varnothing)| = \nu_n(E) < \varepsilon.$$

Hence $\{\nu_n\}$ is uniformly absolutely continuous with respect to μ, as claimed. Since $\{\nu_n\}$ converges setwise to ν, by Theorem 18.25, ν is a finite measure that is absolutely continuous with respect to μ, as claimed. □

Theorem 18.26

Theorem 18.26. Nikodym.
Let $(X, \mathcal{M})$ be a measurable space and $\{\nu_n\}$ a sequence of finite measures on $\mathcal{M}$ which converges setwise on $\mathcal{M}$ to the set function ν. Assume $\{\nu_n(X)\}$ is bounded. Then ν is a measure on $\mathcal{M}$.

Proof. For $E \in \mathcal{M}$, define $\mu(E) = \sum_{n=1}^{\infty} \frac{1}{2^n} \nu_n(E)$. By Exercise 18.63, μ is a finite measure on $\mathcal{M}$. If $\mu(E) = 0$ then each $\nu_n(E) = 0$ and so each ν_n is absolutely continuous with respect to μ. Since $\{\nu_n(X)\}$ is bounded and converges setwise to ν, then by the Vitali-Hahn-Saks Theorem, ν is a measure on $\mathcal{M}$. In fact, ν is finite and absolutely continuous with respect to μ. □