

Real Analysis

Chapter 18. Integration Over General Measure Spaces

18.5. The Nikodym Metric Space: The Vitali-Hahn-Saks Theorem—Proofs of Theorems

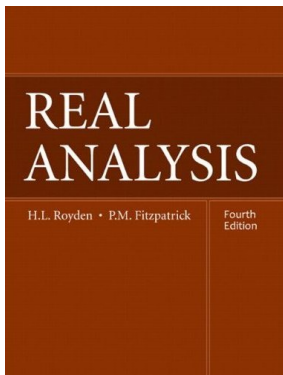


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Lemma 18.5.A

Lemma 18.5.A. The map $\rho_\mu : \mathcal{M}/\simeq \times \mathcal{M}/\simeq \rightarrow \mathbb{R}$ is well-defined and is a metric on \mathcal{M}/\simeq .

Proof. First, we show ρ_μ is well-defined. Let $A_1 \simeq A_2$ and $B_1 \simeq B_2$; that is, $\mu(A_1 \triangle A_2) = \mu(B_1 \triangle B_2) = 0$. So

$$\begin{aligned} \rho_\mu([A_1], [B_2]) &= \mu(A_1 \triangle B_1) \\ &= \mu((A_1 \triangle A_2) \triangle (A_2 \triangle B_1)) \text{ since } (A \triangle B) \triangle (B \triangle C) = A \triangle C \\ &= \nu((A_1 \triangle A_2) \triangle ((A_2 \triangle B_2) \triangle (B_2 \triangle B_1))) \\ &\quad \text{since } (A \triangle B) \triangle (B \triangle C) = A \triangle C. \end{aligned}$$

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Now $\mu(B_1 \triangle B_2) = \mu(B_2 \triangle B_1) = 0$, so $(A \triangle B_2) \triangle (B_2 \triangle B_1)$ consists of almost all elements of $A_1 \triangle B_2$ along with some subset of $B_2 \triangle B_1$ (which must have μ -measure 0), so that $\mu((A_2 \triangle B_2) \triangle (B_2 \triangle B_1)) = \mu(A_2 \triangle B_2)$.

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Lemma 18.5.A (continued 1)

Proof (continued). Similarly, since $\mu(A_1 \triangle A_2) = 0$,

$$\begin{aligned} & \mu((A_1 \triangle A_2) \triangle ((A_2 \triangle B_2) \triangle (B_2 \triangle B_1))) \\ &= \mu((A_2 \triangle B_2) \triangle (B_2 \triangle B_1)) = \mu(A_2 \triangle B_2). \end{aligned}$$

Hence, $\rho_\mu([A_1], [B_1]) = \mu(A_2 \triangle B_2) = \rho_\mu([A_2], [B_2])$ and ρ_μ is well-defined.

We now check the definition of “metric” (see Section 9.1). Let $[A], [B], [C] \in \mathcal{M}/\simeq$.

- (i) $\rho_\mu([A], [B]) = \mu(A \triangle B) \geq 0$,
- (ii) $\rho_\mu([A], [B]) = 0$ if and only if $\mu(A \triangle B) = 0$ if and only if $A \simeq B$ if and only if $[A] = [B]$,
- (iii) $\rho_\mu([A], [B]) = \mu(A \triangle B) = \mu(B \triangle A) = \rho_\mu([B], [A])$,
- (iv) $\rho_\mu([A], [B]) + \rho_\mu([B], [C])$
 $= \mu(A \triangle B) + \mu(B \triangle C) \geq \mu((A \triangle B) \cup (B \triangle C))$
 by subadditivity

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Proof (continued). Similarly, since $\mu(A_1 \triangle A_2) = 0$,

$$\begin{aligned} & \mu((A_1 \triangle A_2) \triangle ((A_2 \triangle B_2) \triangle (B_2 \triangle B_1))) \\ &= \mu((A_2 \triangle B_2) \triangle (B_2 \triangle B_1)) = \mu(A_2 \triangle B_2). \end{aligned}$$

Hence, $\rho_\mu([A_1], [B_1]) = \mu(A_2 \triangle B_2) = \rho_\mu([A_2], [B_2])$ and ρ_μ is well-defined.

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- (ii) $\rho_\mu([A], [B]) = 0$ if and only if $\mu(A \triangle B) = 0$ if and only if $A \simeq B$ if and only if $[A] = [B]$,
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Lemma 18.5.A (continued 2)

Proof (continued).

$$\begin{aligned}
 &\geq \mu((A\Delta B)\Delta(B\Delta C)) \text{ by monotonicity since} \\
 &\quad (A\Delta B)\Delta(B\Delta C) = ((A\Delta B) \setminus (B\Delta C)) \cup ((B\Delta C) \setminus (A\Delta B)) \\
 &\quad \subset (A\Delta B) \cup (B\Delta C) \\
 &= \mu(A\Delta C) \text{ since } (A\Delta B)\Delta(B\Delta C) = A\Delta C \\
 &= \rho_\mu([A], [C]),
 \end{aligned}$$

and the Triangle Inequality holds.

Therefore ρ_μ is a metric. □

Theorem 18.21

Theorem 18.21. Let (X, \mathcal{M}, μ) be a finite measure space. Then the Nikodym metric space (\mathcal{M}, ρ_μ) is complete; that is, every Cauchy sequence converges.

Proof. For $A, B \in \mathcal{M}$, we have $\chi_{A\Delta B} = |\chi_A - \chi_B|$, so

$$\mu(A\Delta B) = \int_X |\chi_A - \chi_B| d\mu. \quad (41)$$

Define $T : \mathcal{M} \rightarrow L^1(X, \mu)$ by $T(E) = \chi_E$ (we need the fact that μ is a finite measure here). Then T is an isometry between the metric spaces since by (41),

$$\rho_\mu(A, B) = \mu(A\Delta B) = \int_X |\chi_A - \chi_B| d\mu = \|T(A) - T(B)\|_1 \text{ for all } A, B \in \mathcal{M}.$$

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Let $\{A_n\}$ be a Cauchy sequence in (\mathcal{M}, ρ_μ) . Then $\{T(A_n)\}$ is a Cauchy sequence in $L^1(X, \mu)$ since T is an isometry.

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Proof (continued). By the Riesz-Fischer Theorem (we actually need the version for measure spaces which is not stated until Section 19.1) there is $f \in L^1(X, \mu)$ such that $\{T(A_n)\} \rightarrow f$ in $L^1(X, \mu)$, and there is a subsequence of $\{T(A_n)\}$ that converges pointwise to f μ -a.e. on X . Since each $T(A_n)$ is a characteristic function and so only takes on the values 0 and 1, if we define A_0 to be the set of points in X at which the pointwise convergent subsequence converges to 1, then the limit function f satisfies $f = \chi_{A_0}$ μ -a.e. on X . Since $\{T(A_n)\}$ converges to $T(A_0) = \chi_{A_0}$ in $L^1(X, \mu)$ then (because T is an isometry) $\{A_n\}$ converges to A_0 in (\mathcal{M}, ρ_μ) . Since $\{A_n\}$ is an arbitrary Cauchy sequence in (\mathcal{M}, ρ_μ) , then (\mathcal{M}, ρ_μ) is complete. □

Theorem 18.21 (continued)

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Lemma 18.22

Lemma 18.22. Let (X, \mathcal{M}, μ) be a finite measure space and ν a finite measure on \mathcal{M} . Let E_0 be a measurable set and $\varepsilon > 0$ and $\delta > 0$ be such that for any measurable set E ,

$$\text{if } \rho_\mu(E, E_0) < \delta \text{ then } |\nu(E) - \nu(E_0)| < \varepsilon/4.$$

Then for any measurable sets A and B ,

$$\text{if } \rho_\mu(A, B) < \delta \text{ then } |\nu(A) - \nu(B)| < \varepsilon.$$

Proof. We first show

$$\text{if } \rho_\mu(A, B) < \delta \text{ then } |\nu(A) - \nu(B)| < \varepsilon/2. \quad (45)$$

Now if $D \subset C$ then $C \Delta D = (C \setminus D) \cup (D \setminus C) = C \setminus D$.

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Now if $D \subset C$ then $C \Delta D = (C \setminus D) \cup (D \setminus C) = C \setminus D$. Let $A \in \mathcal{M}$ and $\rho_\mu(A, \emptyset) = \mu(A \Delta \emptyset) = \mu(A) < \delta$. We have

$$(E_0 \setminus A) \Delta E_0 = (E_0 \setminus (E_0 \setminus A)) \cup (E_0 \setminus (E_0 \setminus A)) = E_0 \setminus (E_0 \setminus A) = E_0 \cap A \subset A.$$

Hence $\rho_\mu(E_0 \setminus A, E_0) = \mu((E_0 \setminus A) \Delta E_0) = \mu(A) < \delta$, and so by hypothesis $|\nu(E_0) - \nu(E_0 \setminus A)| = \nu(E_0) - \nu(E_0 \setminus A) < \varepsilon/4$.

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$$(E_0 \setminus A) \Delta E_0 = (E_0 \setminus (E_0 \setminus A)) \cup (E_0 \setminus (E_0 \setminus A)) = E_0 \setminus (E_0 \setminus A) = E_0 \cap A \subset A.$$

Hence $\rho_\mu(E_0 \setminus A, E_0) = \mu((E_0 \setminus A) \Delta E_0) = \mu(A) < \delta$, and so by hypothesis $|\nu(E_0) - \nu(E_0 \setminus A)| = \nu(E_0) - \nu(E_0 \setminus A) < \varepsilon/4$.

Lemma 18.22 (continued 1)

Proof (continued). By the excision property (Proposition 17.1), since ν is a finite measure,

$$\nu(A \cap E_0) = \nu(E_0 \setminus (E_0 \setminus A)) = \nu(E_0) - \nu(E_0 \setminus A) < \varepsilon/4.$$

Now observe that

$$\begin{aligned} E_0 \Delta (E_0 \cup (A \setminus E_0)) &= (E_0 \setminus (E_0 \cup (A \setminus E_0))) \cup ((E_0 \cup (A \setminus E_0)) \setminus E_0) \\ &= \emptyset \cup ((E_0 \cup (A \setminus E_0)) \setminus E_0) = (E_0 \cup (A \setminus E_0)) \setminus E_0 = A \setminus E_0 \subset A. \end{aligned}$$

So

$$\rho_\mu(E_0, E_0 \cup (A \setminus E_0)) = \mu(E_0 \Delta (E_0 \cup (A \setminus E_0))) = \mu(A \setminus E_0) \leq \mu(A) < \delta.$$

Thus, again by the excision property, and hypothesis

$$\begin{aligned} \nu(A \setminus E_0) &= \nu((E_0 \cup (A \setminus E_0)) \setminus E_0) = |\nu(E_0 \cup (A \setminus E_0)) - \nu(E_0)| \\ &= \nu(E_0 \cup (A \setminus E_0)) - \nu(E_0) < \varepsilon/4. \end{aligned}$$

Lemma 18.22 (continued 1)

Proof (continued). By the excision property (Proposition 17.1), since ν is a finite measure,

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So

$$\rho_\mu(E_0, E_0 \cup (A \setminus E_0)) = \mu(E_0 \Delta (E_0 \cup (A \setminus E_0))) = \mu(A \setminus E_0) \leq \mu(A) < \delta.$$

Thus, again by the excision property, and hypothesis

$$\begin{aligned} \nu(A \setminus E_0) &= \nu((E_0 \cup (A \setminus E_0)) \setminus E_0) = |\nu(E_0 \cup (A \setminus E_0)) - \nu(E_0)| \\ &= \nu(E_0 \cup (A \setminus E_0)) - \nu(E_0) < \varepsilon/4. \end{aligned}$$

Lemma 18.22 (continued 2)

Proof (continued). Therefore

$$\begin{aligned} \nu(A) &= \nu((A \cap E_0) \cup (A \setminus E_0)) \\ &= \nu(A \cap E_0) + \nu(A \setminus E_0) \text{ by additivity} \\ &< \varepsilon/4 + \varepsilon/4 = \varepsilon/2. \end{aligned}$$

So (45) holds.

Now let $A, B \in \mathcal{M}$. Then

$$\begin{aligned} \nu(A) - \nu(B) &= \nu((A \cap B) \cup (A \setminus B)) - \nu((A \cap B) \cup (B \setminus A)) \\ &= \nu(A \cap B) + \nu(A \setminus B) - \nu(A \cap B) - \nu(B \setminus A) \text{ by additivity} \\ &= \nu(A \setminus B) - \nu(B \setminus A), \end{aligned}$$

$$\text{so } |\nu(A) - \nu(B)| = |\nu(A \setminus B) - \nu(B \setminus A)| \leq |\nu(A \setminus B)| + |\nu(B \setminus A)|.$$

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Now let $A, B \in \mathcal{M}$. Then

$$\begin{aligned} \nu(A) - \nu(B) &= \nu((A \cap B) \cup (A \setminus B)) - \nu((A \cap B) \cup (B \setminus A)) \\ &= \nu(A \cap B) + \nu(A \setminus B) - \nu(A \cap B) - \nu(B \setminus A) \text{ by additivity} \\ &= \nu(A \setminus B) - \nu(B \setminus A), \end{aligned}$$

$$\text{so } |\nu(A) - \nu(B)| = |\nu(A \setminus B) - \nu(B \setminus A)| \leq |\nu(A \setminus B)| + |\nu(B \setminus A)|.$$

Lemma 18.22 (continued 3)

Lemma 18.22. Let (X, \mathcal{M}, μ) be a finite measure space and ν a finite measure on \mathcal{M} . Let E_0 be a measurable set and $\varepsilon > 0$ and $\delta > 0$ be such that for any measurable set E ,

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Then for any measurable sets A and B ,

$$\text{if } \rho_\mu(A, B) < \delta \text{ then } |\nu(A) - \nu(B)| < \varepsilon.$$

Proof (continued). Suppose $\rho_\mu(A, B) = \mu(A \Delta B) < \delta$. Since $A \setminus B, B \setminus A \subset A \Delta B$ then $\rho_\mu(A \setminus B, \emptyset) = \mu(A \setminus B) \leq \mu(A \Delta B) < \delta$ and so $\rho_\mu(B \setminus A, \emptyset) = \mu(B \setminus A) \leq \nu(A \Delta B) < \delta$ by monotonicity (Proposition 17.1). So by (45), we have $\nu(A \setminus B) < \varepsilon/2$ and $\nu(B \setminus A) < \varepsilon/2$. Hence, $|\nu(A) - \nu(B)| \leq |\nu(A \setminus B)| + |\nu(B \setminus A)| < \varepsilon$, as claimed. \square

Proposition 18.23

Proposition 18.23. Let (X, \mathcal{M}, μ) be a finite measure space and ν a finite measure on \mathcal{M} that is absolutely continuous with respect to μ . Then ν induces a properly defined (i.e., “well-defined”), uniformly continuous function on the Nikodym metric space associated with (X, \mathcal{M}, μ) .

Proof. Since finite measure space ν is absolutely continuous with respect to μ then, by Proposition 18.19, for every $\varepsilon > 0$ there is $\delta > 0$ such that for any $E \in \mathcal{M}$, if $\mu(E) < \delta$ then $\nu(E) < \varepsilon$. Since $\mu(E) = \rho_\mu(E, \emptyset)$, this means that $\nu : \mathcal{M} \rightarrow \mathbb{R}$ is continuous at $E_0 = \emptyset$ (where ν maps metric space (\mathcal{M}, ρ_μ) to metric space $(\mathbb{R}, |\cdot|)$). By Lemma 18.22, ν is uniformly continuous on (\mathcal{M}, ρ_μ) . □

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Theorem 18.25

Theorem 18.25. Let (X, \mathcal{M}, μ) be a finite measure space and $\{\nu_n\}$ a sequence of finite measures on \mathcal{M} that is uniformly absolutely continuous with respect to μ . If $\{\nu_n\}$ converges setwise on \mathcal{M} to ν , then ν is a measure of \mathcal{M} that is absolutely continuous with respect to μ .

Proof. Since each ν_n is nonnegative then ν is nonnegative. Let $\{E_k\}_{k=1}^m$ be disjoint. Then

$$\begin{aligned} \nu\left(\bigcup_{k=1}^m E_k\right) &= \lim_{n \rightarrow \infty} \nu_n\left(\bigcup_{k=1}^m E_k\right) = \lim_{n \rightarrow \infty} (\nu_n(E_1) + \nu_n(E_2) + \cdots + \nu_n(E_m)) \\ &= \lim_{n \rightarrow \infty} \nu_n(E_1) + \lim_{n \rightarrow \infty} \nu_n(E_2) + \cdots + \lim_{n \rightarrow \infty} \nu_n(E_m) = \nu(E_1) + \nu(E_2) + \cdots + \nu(E_m) \end{aligned}$$

and so ν is finite additive.

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Proof. Since each ν_n is nonnegative then ν is nonnegative. Let $\{E_k\}_{k=1}^m$ be disjoint. Then

$$\begin{aligned} \nu(\cup_{k=1}^m E_k) &= \lim_{n \rightarrow \infty} \nu_n(\cup_{k=1}^m E_k) = \lim_{n \rightarrow \infty} (\nu_n(E_1) + \nu_n(E_2) + \cdots + \nu_n(E_m)) \\ &= \lim_{n \rightarrow \infty} \nu_n(E_1) + \lim_{n \rightarrow \infty} \nu_n(E_2) + \cdots + \lim_{n \rightarrow \infty} \nu_n(E_m) = \nu(E_1) + \nu(E_2) + \cdots + \nu(E_m) \end{aligned}$$

and so ν is finite additive. If $A \subset B$ then $\nu_n(A) \leq \nu_n(B)$, so $\nu(A) = \lim_{n \rightarrow \infty} \nu_n(A) \leq \lim_{n \rightarrow \infty} \nu_n(B) = \nu(B)$ and so ν is monotone. For countable additivity, consider disjoint $\{E_k\}_{k=1}^{\infty}$. If $\nu(E_k) = \infty$ for some E_k , then $\nu(\cup_{k=1}^{\infty} E_k) \geq \nu(E_k) = \infty = \sum_{k=1}^{\infty} \nu(E_k)$ and so countable additivity holds. So we can assume without loss of generality that $\nu(E_k) < \infty$ for all $k \in \mathbb{N}$.

Theorem 18.25

Theorem 18.25. Let (X, \mathcal{M}, μ) be a finite measure space and $\{\nu_n\}$ a sequence of finite measures on \mathcal{M} that is uniformly absolutely continuous with respect to μ . If $\{\nu_n\}$ converges setwise on \mathcal{M} to ν , then ν is a measure of \mathcal{M} that is absolutely continuous with respect to μ .

Proof. Since each ν_n is nonnegative then ν is nonnegative. Let $\{E_k\}_{k=1}^m$ be disjoint. Then

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Theorem 18.25 (continued 1)

Proof (continued). By finite additivity, for each $n \in \mathbb{N}$

$$\nu(\cup_{k=1}^{\infty} E_k) = \sum_{k=1}^n \nu(E_k) + \nu(\cup_{k=n+1}^{\infty} E_k). \quad (47)$$

Let $\varepsilon > 0$. Since $\{\nu_n\}$ is uniformly absolutely continuous with respect to μ , there is $\delta > 0$ such that for $E \in \mathcal{M}$ and for all $n \in \mathbb{N}$,

$$\text{if } \mu(E) < \delta \text{ then } \nu_n(E) < \varepsilon/2. \quad (48)$$

Since $\nu(E) = \lim_{n \rightarrow \infty} \nu_n(E)$ for all $E \in \mathcal{M}$, then $\nu_n(E) < \varepsilon/2$ implies $\nu(E) \leq \varepsilon/2 < \varepsilon$. Since $\mu(X) < \infty$ and μ is countably additive then $\mu(\cup_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty} \mu(E_k) < \infty$ and so there is $N \in \mathbb{N}$ for which $\mu(\cup_{k=N+1}^{\infty} E_k) = \sum_{k=N+1}^{\infty} \mu(E_k) < \delta$ (since the “tail” of a convergent series must be small) and so $\nu(\cup_{k=N+1}^{\infty} E_k) < \varepsilon$. So from (47),

$$0 \leq \nu(\cup_{k=1}^{\infty} E_k) - \sum_{k=1}^N \nu(E_k) = \nu(\cup_{k=N+1}^{\infty} E_k) < \varepsilon.$$

Theorem 18.25 (continued 1)

Proof (continued). By finite additivity, for each $n \in \mathbb{N}$

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Theorem 18.25 (continued 2)

Theorem 18.25. Let (X, \mathcal{M}, μ) be a finite measure space and $\{\nu_n\}$ a sequence of finite measures on \mathcal{M} that is uniformly absolutely continuous with respect to μ . If $\{\nu_n\}$ converges setwise on \mathcal{M} to ν , then ν is a measure of \mathcal{M} that is absolutely continuous with respect to μ .

Proof (continued). Therefore

$$\lim_{N \rightarrow \infty} \left(\sum_{k=1}^N \nu(E_k) \right) = \sum_{k=1}^{\infty} \nu(E_k) = \nu \left(\bigcup_{k=1}^{\infty} E_k \right),$$

as claimed. So ν is a measure on \mathcal{M} .

From (48) (based on the uniform absolute continuity of $\{\nu_n\}$ with respect to μ) we see that if $\mu(E) = 0$ then $\nu_n(E) = 0$ for all $n \in \mathbb{N}$ and so $\nu(E) = \lim_{n \rightarrow \infty} \nu_n(E) = 0$. So ν is absolutely continuous with respect to μ . □

Theorem 18.25 (continued 2)

Theorem 18.25. Let (X, \mathcal{M}, μ) be a finite measure space and $\{\nu_n\}$ a sequence of finite measures on \mathcal{M} that is uniformly absolutely continuous with respect to μ . If $\{\nu_n\}$ converges setwise on \mathcal{M} to ν , then ν is a measure of \mathcal{M} that is absolutely continuous with respect to μ .

Proof (continued). Therefore

$$\lim_{N \rightarrow \infty} \left(\sum_{k=1}^N \nu(E_k) \right) = \sum_{k=1}^{\infty} \nu(E_k) = \nu \left(\bigcup_{k=1}^{\infty} E_k \right),$$

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From (48) (based on the uniform absolute continuity of $\{\nu_n\}$ with respect to μ) we see that if $\mu(E) = 0$ then $\nu_n(E) = 0$ for all $n \in \mathbb{N}$ and so $\nu(E) = \lim_{n \rightarrow \infty} \nu_n(E) = 0$. So ν is absolutely continuous with respect to μ . □

The Vitali-Hahn-Saks Theorem

The Vitali-Hahn-Saks Theorem.

Let (X, \mathcal{M}, μ) be a finite measure space and $\{\nu_n\}$ a sequence of finite measures on \mathcal{M} , each of which is absolutely continuous with respect to μ . Suppose that $\{\nu_n(X)\}$ is bounded and $\{\nu_n\}$ converges setwise on \mathcal{M} to ν . Then the sequence $\{\nu_n\}$ is uniformly continuous with respect to μ . Moreover, ν is a finite measure on \mathcal{M} that is absolutely continuous with respect to μ .

Proof. By Proposition 18.23, $\{\nu_n\}$ induces a sequence of (uniformly) continuous functions on the Nikodym metric space where $\nu_n : \mathcal{M} \rightarrow \mathbb{R}$ (and the domain of ν_n is the metric space (\mathcal{M}, ρ_μ)). Since $\{\nu_n(X)\}$ is bounded for ν is finite on \mathcal{M} . By Theorem 18.21, (\mathcal{M}, ρ_μ) is a complete metric space.

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Proof. By Proposition 18.23, $\{\nu_n\}$ induces a sequence of (uniformly) continuous functions on the Nikodym metric space where $\nu_n : \mathcal{M} \rightarrow \mathbb{R}$ (and the domain of ν_n is the metric space (\mathcal{M}, ρ_μ)). Since $\{\nu_n(X)\}$ is bounded for ν is finite on \mathcal{M} . By Theorem 18.21, (\mathcal{M}, ρ_μ) is a complete metric space.

The Vitali-Hahn-Saks Theorem (continued)

Proof (continued). So by Theorem 10.7 (a consequence of the Baire Category Theorem), there is a set $E_0 \in \mathcal{M}$ for which the sequence $\{\nu_n : \mathcal{M} \rightarrow \mathbb{R}\}$ is equicontinuous at E_0 ; that is, for each $\varepsilon > 0$ there is $\delta > 0$ such that for each measurable set E and $n \in \mathbb{N}$,

$$\text{if } \rho_\mu(E, E_0) < \delta \text{ then } |\nu_n(E) - \nu_n(E_0)| < \varepsilon.$$

Since each ν_n is finite, Lemma 18.22 implies that for each $\varepsilon > 0$ there is $\delta > 0$ such that for each $E \in \mathcal{M}$ and each $n \in \mathbb{N}$

$$\text{if } \rho_\mu(E, \emptyset) = \mu(E) < \delta \text{ then } |\nu_n(E) - \nu_n(\emptyset)| = \nu_n(E) < \varepsilon.$$

Hence $\{\nu_n\}$ is uniformly absolutely continuous with respect to μ , as claimed. Since $\{\nu_n\}$ converges setwise to ν , by Theorem 18.25, ν is a finite measure that is absolutely continuous with respect to μ , as claimed. \square

The Vitali-Hahn-Saks Theorem (continued)

Proof (continued). So by Theorem 10.7 (a consequence of the Baire Category Theorem), there is a set $E_0 \in \mathcal{M}$ for which the sequence $\{\nu_n : \mathcal{M} \rightarrow \mathbb{R}\}$ is equicontinuous at E_0 ; that is, for each $\varepsilon > 0$ there is $\delta > 0$ such that for each measurable set E and $n \in \mathbb{N}$,

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Since each ν_n is finite, Lemma 18.22 implies that for each $\varepsilon > 0$ there is $\delta > 0$ such that for each $E \in \mathcal{M}$ and each $n \in \mathbb{N}$

$$\text{if } \rho_\mu(E, \emptyset) = \mu(E) < \delta \text{ then } |\nu_n(E) - \nu_n(\emptyset)| = \nu_n(E) < \varepsilon.$$

Hence $\{\nu_n\}$ is uniformly absolutely continuous with respect to μ , as claimed. Since $\{\nu_n\}$ converges setwise to ν , by Theorem 18.25, ν is a finite measure that is absolutely continuous with respect to μ , as claimed. \square

Theorem 18.26

Theorem 18.26. Nikodym.

Let (X, \mathcal{M}) be a measurable space and $\{\nu_n\}$ a sequence of finite measures on \mathcal{M} which converges setwise on \mathcal{M} to the set function ν . Assume $\{\nu_n(X)\}$ is bounded. Then ν is a measure on \mathcal{M} .

Proof. For $E \in \mathcal{M}$, define $\mu(E) = \sum_{n=1}^{\infty} \frac{1}{2^n} \nu_n(E)$. By Exercise 18.63, μ is a finite measure on \mathcal{M} . If $\mu(E) = 0$ then each $\nu_n(E) = 0$ and so each ν_n is absolutely continuous with respect to μ . Since $\{\nu_n(X)\}$ is bounded and converges setwise to ν , then by the Vitali-Hahn-Saks Theorem, ν is a measure on \mathcal{M} . In fact, ν is finite and absolutely continuous with respect to μ . □

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Let (X, \mathcal{M}) be a measurable space and $\{\nu_n\}$ a sequence of finite measures on \mathcal{M} which converges setwise on \mathcal{M} to the set function ν . Assume $\{\nu_n(X)\}$ is bounded. Then ν is a measure on \mathcal{M} .

Proof. For $E \in \mathcal{M}$, define $\mu(E) = \sum_{n=1}^{\infty} \frac{1}{2^n} \nu_n(E)$. By Exercise 18.63, μ is a finite measure on \mathcal{M} . If $\mu(E) = 0$ then each $\nu_n(E) = 0$ and so each ν_n is absolutely continuous with respect to μ . Since $\{\nu_n(X)\}$ is bounded and converges setwise to ν , then by the Vitali-Hahn-Saks Theorem, ν is a measure on \mathcal{M} . In fact, ν is finite and absolutely continuous with respect to μ . □