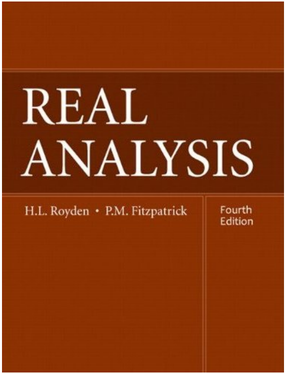
Real Analysis

Chapter 19. General L^p Spaces: Completeness, Duality, and Weak Convergence

19.1. The Completeness of $L^p(X, \mu)$, $1 \leq p \leq \infty$—Proofs of Theorems



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Theorem 19.1

Theorem 19.1. Let $(X, \mathcal{M}, \mu)$ be a measure space, $1 \le p \le \infty$, and q the conjugate of p (that is, $\frac{1}{p} + \frac{1}{q} = 1$). If $f \in L^p(X, \mu)$ and $g \in L^q(X, \mu)$, then the product $fg \in L^1(X, \mu)$ and:

(i) Hölder's Inequality. $\int_X |fg|\, d\mu = \|fg\|_1 \le \|f\|_p \|g\|_q$. Moreover, if $f \ne 0$, the function $f^* = \|f\|_p^{1-p} \mathrm{sgn}(f)|F|^{p-1} \in L^q(X, \mu)$, $\int_X ff^*\, d\mu = \|f\|_p$ and $\|f^*\|_q = 1$.

(ii) Minkowski's Inequality. For $1 \le p \le \infty$ and $f, g \in L^p(X, \mu)$, $\|f + g\|_p \le \|f\|_p + \|g\|_p$. Therefore $L^p(X, \mu)$ is a normed linear space.

(iii) The Cauchy-Schwarz Inequality. Let f and g be measurable functions on X for which f^2 and g^2 are integrable over X. Then their product fg also is integrable over X and $\int_E |fg|\, d\mu \le \sqrt{\int_X f^2\, d\mu}\sqrt{\int_X g^2\, d\mu}$.

Proposition 19.1 (continued 1)

Proof (continued). Once we establish Hólder's Inequality, the claim that $f \in L^p(X,\mu)$ and $g \in L^1(X,\mu)$ implu $fg \in L^1(X,\mu)$ will then follow.
(i) If $p = 1$ and $q = \infty$ then

$$\begin{aligned}\int_X |fg|\,d\mu &\leq \int_X \|g\|_\infty |f\|\,d\mu \text{ by monotonicity, Lemma 18.2.A}\\ &= \|g\|_\infty \int_X |f|\,d\mu \text{ by linearity, Lemma 18.2.A}\\ &= \|f\|_1 \|g\|_\infty.\end{aligned}$$

Also, for $p = 1$ and $q = \infty$, if $f \neq 0$ then

$$|f^*| = |\|f\|_p^{p-1}\mathrm{sgn}(f)|f|^{p-1}| = \|f\|_1^0 |f^0| = 1,$$

so $\|f^*\|_\infty = 1$ and $f^* \in L^q(X,\mu)$. In addition, $\int_X ff^*\,d\mu = \int_X f\|f\|_p^{1-p}\mathrm{sgn}(f)|f|^{p-1}\,d\mu = \int_X |f|\|f\|_1^0|f|^0\,d\mu = \int_X |f|\,d\mu = \|f\|_1$, so that Hölder's Inequality holds for $p = 1$.

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(i) If $p = 1$ and $q = \infty$ then

$$\begin{aligned} \int_X |fg| \, d\mu &\leq \int_X \|g\|_\infty |f\| \, d\mu \text{ by monotonicity, Lemma 18.2.A} \\ &= \|g\|_\infty \int_X |f| \, d\mu \text{ by linearity, Lemma 18.2.A} \\ &= \|f\|_1 \|g\|_\infty. \end{aligned}$$

Also, for $p = 1$ and $q = \infty$, if $f \neq 0$ then

$$|f^*| = |\|f\|_p^{p-1} \text{sgn}(f) |f|^{p-1}| = \|f\|_1^0 |f^0| = 1,$$

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Proposition 19.1 (continued 2)

Proof (continued). Now consider $p > 1$. Young's Inequality gives for $a, b \in \mathbb{R}$, $ab \le \frac{1}{p}a^p + \frac{1}{q}b^q$. Define $\alpha = \int_X |f|^p \, d\mu$ and $\beta = \int_X |g|^q \, d\mu$. If either $\alpha = 0$ or $\beta = 0$ then either $\|f\|_p = 0$ or $\|g\|_q = 0$ (respectively) and so either $f = 0$ a.e. or $g = 0$ a.e. In either case, $\int_X |fg| \, d\mu = 0$ and Hölder's Inequality holds. So we can without loss of generality assume α and β are both positive. Since $f \in L^p(X, \mu)$ and $g \in L^q(X, \mu)$ then f and g are both finite μ-a.e. by Proposition 18.9. If $f(x)$ and $g(x)$ are finite, substitute $|f(x)|/\alpha^{1/p}$ for a and $|g(x)|/\beta^{1/q}$ for b in Young's Inequality to conclude that

$$ab = \frac{1}{\alpha^{1/p}\beta^{1/q}}|f(x)g(x)| \le \frac{1}{p}a^p + \frac{1}{q}\beta^q = \frac{1}{p}\frac{1}{\alpha}|f(x)|^p + \frac{1}{q}\frac{1}{\beta}|g(x)|^q$$

for almost all $x \in X$.

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for almost all $x \in X$.

Proposition 19.1 (continued 3)

Proof (continued). Integrating over X gives

$$\frac{1}{\alpha^{1/p}\beta^{1/q}}\int_X |fg|\,d\mu \leq \frac{1}{p}\frac{1}{\alpha}\int_X |f|^p + \frac{1}{q}\frac{1}{\beta}\int_X |g|^q\,d\mu \text{ by linearity,}$$

Prop. 18.11, and monotonicity, Lemma 18.2.A(ii)

$$= \frac{1}{p}\frac{1}{\alpha}\alpha + \frac{1}{q}\frac{1}{\beta}\beta = \frac{1}{p} + \frac{1}{q} = 1,$$

or $\int_X |fg|\,d\mu = \|fg\|_1 \leq \alpha^{1/P}\beta^{1/q} = \|f\|_p\|g\|_q$, as claimed. So Hölder's Inequality holds for $1 \leq p < \infty$.

(ii) We commented above that if $f, g \in L^p(X,\mu)$ then $f + g \in L^p(X,\mu)$. So, by Hölder's Inequality (the "Moreover" part), $(f+g)^* \in L^q(X,\mu)$.

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or $\int_X |fg|\,d\mu = \|fg\|_1 \le \alpha^{1/P}\beta^{1/q} = \|f\|_p\|g\|_q$, as claimed. So Hölder's Inequality holds for $1 \le p < \infty$.

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Proposition 19.1 (continued 4)

Proof (continued). Therefore,

$$\begin{aligned} \|f+g\|_p &= \int_X (f+g)(f+g)^*\, d\mu \text{ by Hölder's Inequality ("Moreover")} \\ &= \int +Xf(f+g)^*\, d\mu + \int_X g(f+g)^*\, d\mu \text{ by linearity, Propositio} \end{aligned}$$

Now $\int_X |f(f+g)^*|\, d\mu \le \|f\|_p \|(f+g)^*\|_q$ by Hölder's Inequality and $f(f+g)^* \le |f(f+g)^*$ on X, so by the Integral Comparison Test,

$$\int_X f(f+g)^*\, d\mu \le \left| \int_X f(f+g)^*\, d\mu \right| \le \int_X |f(f+g)^*|\, d\mu \le \|f\|_p \|(f+g)^*\|_q.$$

Similarly $\int_X g(f+g)^*\, d\mu \le \|g\|_p \|(f+g)^*\|_q$.

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$$\begin{aligned}\|f+g\|_p &= \int_X f(f+g)^*\,d\mu + \int_X f(f+g)^*\,d\mu \\ &\leq \|f\|_p\|(f+g)^*\|_q + \|g\|_p\|(f+g)^*\|_q \\ &= \|f\|_p + \|g\|_q \text{ since } \|(f+g)^*\|_q = 1 \text{ by Hölder's Inequality} \\ &\quad \text{the ``Moreover'' part).}\end{aligned}$$

So Minkowski's Inequality holds, as claimed.

(iii) With $p = q = 2$ in Hölder's Inequality, we get the Cauchy-Schwarz Inequality. □

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Corollary 19.3

Corollary 19.3. Let $(X, \mathcal{M}, \mu)$ be a measure space and $1 < p \leq \infty$. If $\{f_n\}$ is a bounded sequence of functions in $L^p(X, \mu)$, then $\{f_n\}$ is uniformly integrable over X.

Proof. Let $M > 0$ be such that $\|f\|_p \leq M$ for $n \in \mathbb{N}$. Define $\gamma = 1$ if $p = \infty$ and $\gamma = (p-1)/p$ if $1 < p < \infty$. By Corollary 19.2, with $p_1 = 1$, $p_2 = p$, and X of Corollary 19.2 replaced with any set E of finite measure in $\mathcal{M}$, we have

$$\|f\|_{p_1} = \|p\|_1 = \int_E |f| \, d\mu \leq c\|f\|_{p_2} \leq Mc = M(\mu(E))^\gamma. \qquad (*)$$

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Let $\varepsilon > 0$. Let $\delta = (\varepsilon/M)^{1/\gamma}$. If $A \subset X$ is μ-measurable with $\mu(A) < \delta$ then from $(*)$ with $E = A$,

$$\int_A |f|\, d\mu \leq M(\mu(A))^\gamma < M((\varepsilon/M))^{1/\gamma})^\gamma = \varepsilon.$$

So $\{f_n\}$ is uniformly integrable over X (by definition), as claimed. □

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Lemma 19.4

Lemma 19.4. Let $(X, \mathcal{M}, \mu)$ be a measure space and $1 \leq p \leq \infty$. Then every rapidly Cauchy sequence in $L^p(X, \mu)$ converges to a function in $L^p(X, \mu)$, both with respect to the $L^p(X, \mu)$ norm and pointwise a.e. in X.

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Let $(X, \mathcal{M}, \mu)$ be a measure space and $1 \leq p \leq \infty$. Then $L^p(X, \mu)$ is a Banach space. Moreover, if a sequence in $L^p(X, \mu)$ converges in $L^p(X, \mu)$ to $f \in L^p(X, \mu)$, then a subsequence converges pointwise a.e. on X to f.

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Theorem 19.5

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