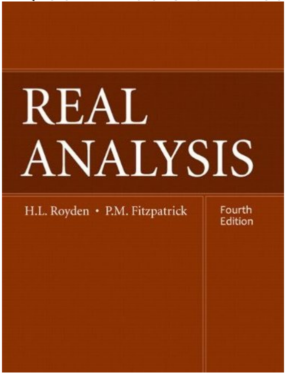
Real Analysis

Chapter 19. General L^p Spaces: Completeness, Duality, and Weak Convergence

19.2. The Reisz Representation Theorem for the Dual of $L^p(X, \mu)$, $1 \leq p < \infty$—Proofs of Theorems



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Lemma 19.6

Lemma 19.6. Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space and let $1 \leq p < \infty$. For f an integrable function over X, suppose there is $M \leq 0$ such that for every simple function g on X that vanishes outside a set of finite measure, we have $\left|\int_X fg\, d\mu\right| \leq M\|g\|_p$. Then $f \in L^q(X, \mu)$ where q is the conjugate of p. Moreover, $\|f\|_q \leq M$.

Proof. First, suppose $p > 1$. Since $|f|$ is a nonnegative function and the measure space is σ-finite, then by the Simple Approximation Theorem (Section 18.1) there is a sequence of simple functions $\{\varphi_n\}$, each of which vanishes outside a set of finite measure, that converges pointwise on X to $|f|$ and $0 \leq \varphi_n \leq |f|$ on E for all $n \in \mathbb{N}$.

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$$\int_X \varphi_n^q\, d\mu \le M^q \text{ for } n \in \mathbb{N}. \qquad (10)$$

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$$\int_X \varphi_n^q\,d\mu \le M^q \text{ for } n \in \mathbb{N}. \qquad (10)$$

Lemma 19.6 (continued 1)

Proof (continued). Fix $n \in \mathbb{N}$ We have

$$\varphi_n^q = \varphi_n \varphi_n^{q-1} \leq |f| \varphi_n^{q-1} = f \operatorname{sgn}(f) \varphi_n^{q-1} \text{ on } X.$$

Define the simple function g_n by $g_n = \operatorname{sgn}(f)\varphi_n^{q-1}$ on X. Then

$$\begin{aligned}
\int_X \varphi_n^q \, d\mu &\leq \in_X |f| \varphi_n^{q-1} \, d\mu \text{ by monotonicity of the integral} \\
&\quad \text{(Theorem 18.12)} \\
&= \left| \int_X |f| \varphi_n^{q-1} \, d\mu \right| \text{ since } |f|\varphi_n^{q-1} \geq 0 \\
&= \left| \int_X f \operatorname{sgn}(f) \varphi_n^{q-1} \, d\mu \right| = \left| \int_x f g_n \, d\mu \right| \\
&\leq M \|g_n\|_p \text{ by the hypothesis of the lemma.} \qquad (12)
\end{aligned}$$

Lemma 19.6 (continued 2)

Proof (continued). Since p and q are conjugates then $p(q-1)=q$ and therefore

$$\begin{aligned}\int_X |g_n|^p\,d\mu &= \int_X (\varphi_n^{q-1})^p\,d\mu \text{ since } g_n = \text{sgn}(f)\varphi_n^{q-1}\\ &= \int_X \varphi_n^q\,d\mu,\end{aligned}$$

so from (12) we have

$$\int_X \varphi_n^q\,d\mu \le M\|g_n\|_p = \left(\int_X \varphi_n^q\,d\mu\right)^{1/p}.$$

Now for each $n \in \mathbb{N}$, φ_n^q is a simple function that vanishes outside a set of finite measure and so is integrable (that is, $\int_X \varphi_n^q\,d\mu < \infty$). So the last inequality implies $\left(\int_X \varphi_n^q\,d\mu\right)^{1/q} \le M$ and $\int_X \varphi_n^q\,d\mu \le M^q$. So, as described above, this is (1) and $n \in \mathbb{N}$ is arbitrary, so the claim holds in the case $p > 1$.

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so from (12) we have

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Now for each $n \in \mathbb{N}$, φ_n^q is a simple function that vanishes outside a set of finite measure and so is integrable (that is, $\int_X \varphi_n^q \, d\mu < \infty$). So the last inequality implies $\left(\int_X \varphi_n^q \, d\mu \right)^{1/q} \leq M$ and $\int_X \varphi_n^q \, d\mu \leq M^q$. So, as described above, this is (1) and $n \in \mathbb{N}$ is arbitrary, so the claim holds in the case $p > 1$.

Lemma 19.6 (continued 3)

Proof (continued). Second, suppose $p = 1$ (so that $q = \infty$). ASSUME M given in the hypotheses is not an essential upper bound of $\|f\|_q$. Then there is some $\varepsilon > 0$ for which the set $X_\varepsilon = \{x \in X \mid |f(x)| > M + \varepsilon\}$ has positive measure. Since X is σ-finite there is a subset $A \subset X_\varepsilon$ with finite positive measure (eliminating the case that X_ε has infinite measure and all its measurable subsets have either measure 0 or ∞). With $g = \chi_A$ (a simple function on X that vanishes outside a set of finite measure) we have

$$\begin{aligned}
\int_X fg\, d\mu &= \int_X f\chi_A\, d\mu = \int_A f\, d\mu \\
&\geq \int_A (M+\varepsilon)\, d\mu \text{ since } f > M+\varepsilon \text{ on } A \\
&= (M+\varepsilon)m(A) = (m+\varepsilon)\|g\|_1 \text{ since } \|g_1\| = \|\chi_A\|_1 = m(A) \\
&> M\|g\|_1,
\end{aligned}$$

CONTRADICTING the hypotheses of the lemma, showing the assumption is false and $\int_X fg\, d\mu \leq M\|g\|_1$, as claimed. □

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The Riesz Representation Theorem

The Riesz Representation Theorem for the Dual of $L^p(X, \mu)$.
Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space, let $1 \leq p < \infty$, and let q be the conjugate of p. For $f \in L^q(X, \mu)$ define $T_f \in (L^p(X, \mu))^*$ as $T_f(g) = \int_X fg\, d\mu$. Then $T : L^q(X, \mu) \to (L^p(X, \mu))^*$, defined as $T(f) = T_f$, is an isometric isomorphism.

Proof. The case $p = 1$ is to be given in Exercise 19.6.

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Proof. The case $p = 1$ is to be given in Exercise 19.6.
Suppose $p > 1$. We first consider the case $\mu(X) < \infty$. Let $S : L^p(X, \mu) \to \mathbb{R}$ be a bounded linear functional. Define set function ν on the collection of measurable sets $\mathcal{M}$ by setting $\nu(E) = S(\chi_E)$ for $E \in \mathcal{M}$. Sine $\mu(X) < \infty$, then each characteristic function of each measurable set is integrable and in $L^p(X, \mu)$, so that $\nu(E)$ is "properly defined." We claim that ν is a signed measure.

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Let $(X,\mathcal{M},\mu)$ be a σ-finite measure space, let $1 \le p < \infty$, and let q be the conjugate of p. For $f \in L^q(X,\mu)$ define $T_f \in (L^p(X,\mu))^*$ as $T_f(g) = \int_X fg\,d\mu$. Then $T : L^q(X,\mu) \to (L^p(X,\mu))^*$, defined as $T(f) = T_f$, is an isometric isomorphism.

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The Riesz Representation Theorem (continued 1)

Proof (continued). First, for $\{E_k\}_{k=1}^{\infty}$ a countable disjoint collection of measurable sets and $E = \cup_{k=1}^{\infty} E_k$ we have by the countable additivity of μ

$$\mu(E) = \mu\left(\cup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mu(E_k) < \infty.$$

So $\lim_{n\to\infty}\left(\sum_{k=n+1}^{\infty} \mu(E_k)\right) = 0$ since the tail of a summable series must go to 0. Consequently

$$\begin{aligned} \lim_{n\to\infty}\left\|\chi_E - \sum_{k=1}^{n} \chi_{E_k}\right\|_p &= \lim_{n\to\infty}\left(\int_X \left|\sum_{k=n+1}^{\infty} \chi_{E_k}\right|^p d\mu\right)^{1/p} \\ &= \lim_{k\to\infty}\left(\int_X \left(\sum_{k=n+1}^{\infty} \chi_{E_k}\right) d\mu\right)^{1/p} \\ &\quad \text{since } \sum_{k=n+1}^{\infty} \chi_{E_k} = 0 \text{ or } 1 \text{ on } X \ldots \end{aligned}$$

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$$\begin{aligned}
\lim_{n\to\infty} \left\| \chi_E - \sum_{k=1}^{n} \chi_{E_k} \right\|_p &= \lim_{n\to\infty} \left(\int_X \left| \sum_{k=n+1}^{\infty} \chi_{E_k} \right|^p d\mu \right)^{1/p} \\
&= \lim_{k\to\infty} \left(\int_X \left(\sum_{k=n+1}^{\infty} \chi_{E_k} \right) d\mu \right)^{1/p} \\
&\quad \text{since } \sum_{k=n+1}^{\infty} \chi_{E_k} = 0 \text{ or } 1 \text{ on } X \ldots
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The Riesz Representation Theorem (continued 2)

Proof (continued).

$$\lim_{n\to\infty}\left\|\chi_E-\sum_{k=1}^{n}\chi_{E_k}\right\|_p = \lim_{n\to\infty}\left(\sum_{k=n+1}^{\infty}\left(\int_X \chi_{E_k}\,d\mu\right)\right)^{1/p}$$

by the Monotone Convergence Theorem

$$= \lim_{n\to\infty}\left(\sum_{k=n+1}^{\infty}\mu(E_k)\right)^{1/p} = 0.$$

Since S is linear and continuous on $L^p(X,\mu)$ (every bounded linear functional is continuous; see Section 8.1) and hence

$$S(\chi_E) = S\left(\chi_{\cup E_k}\right) = S\left(\sum_{k=1}^{\infty}\chi_{E_k}\right) = \sum_{k=1}^{\infty}S(\chi_{E_k}),$$

so $\nu(E)=\sum_{k=1}^{\infty}\nu(E_k)$ and ν is countable additive.

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Proof (continued).

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Proof (continued). Since functional S does not take on the values $\pm\infty$ and S maps the 0 function (namely, $\chi_\varnothing$) to 0, then to show that ν is a signed measure, we just need to show that $\sum_{k=1}^\infty \nu(E_k)$ above converges absolutely)see Section 17.2). For each $k \in \mathbb{N}$, set $c_k = \text{sgn}(S(\chi_{E_k}))$ (notice $S(\chi_{E_k}) \in \mathbb{R}$), then we have

$$\sum_{k=1}^\infty S(c_k\chi_{E_k}) = \sum_{k=1}^\infty c_k S(\chi_{E_k}) = \sum_{k=1}^\infty |S(\chi_{E_k})| = \sum_{k=1}^\infty |\nu(E_k)|$$

since $\nu(E) = S(\chi_E)$ for $E \in \mathcal{M}$. Also, for all $x \in E$ we have

$$\left|\sum_{k=1}^\infty c_k\chi_{E_k}(x)\right| \le \sum_{k=1}^\infty |c_k\chi_{E_k}(x)| = \sum_{k=1}^\infty \chi_{E_k} = \chi_E(x),$$

and since $\chi_E \in L^p(X, \mu)$ then $\sum_{k=1}^\infty c_k\chi_{E_k}(x) \in L^p(X, \mu)$.

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since $\nu(E) = S(\chi_E)$ for $E \in \mathcal{M}$. Also, for all $x \in E$ we have

$$\left| \sum_{k=1}^\infty c_k \chi_{E_k}(x) \right| \leq \sum_{k=1}^\infty |c_k \chi_{E_k}(x)| = \sum_{k=1}^\infty \chi_{E_k} = \chi_E(x),$$

and since $\chi_E \in L^p(X, \mu)$ then $\sum_{k=1}^\infty c_k \chi_{E_k}(x) \in L^p(X, \mu)$.

The Riesz Representation Theorem (continued 4)

Proof (continued). As above,

$$\lim_{n\to\infty} \left\| \sum_{k=1}^{\infty} c_k \chi_{E_k} - \sum_{k=1}^{n} c_k \chi_{E_k} \right\|_p$$

$$\begin{aligned}
&= lim_{n\to\infty} \left\| \sum_{k=n+1}^{\infty} c_k \chi_{E_k} \right\|_p = \lim_{n\to\infty} \left(\int_X \left| \sum_{k=n+1}^{\infty} c_k \chi_{E_k} \right|^p d\mu \right)^{1/p} \\
&= \lim_{n\to\infty} \left(\int_X \sum_{k=n+1}^{\infty} |c_k \chi_{E_k}|^p \, d\mu \right)^{1/p} \quad \text{since the } E_k \text{ are disjoint} \\
&= \lim_{n\to\infty} \left(\int_X \sum_{k=n+1}^{\infty} \chi_{E_k} \, d\mu \right)^{1/p} \quad \text{since } c_k \chi_{E_k} \in \{-1, 0, 1\} \\
&= 0 \text{ as shown above.}
\end{aligned}$$

The Riesz Representation Theorem (continued 5)

Proof (continued). Since S is linear and continuous on $L^p(X,\mu)$ then

$$S\left(\sum_{k=1}^{\infty} c_k \chi_{E_k}\right) = \sum_{k=1}^{\infty} S(c_k \chi_{E_k})$$

$$= \sum_{k=1}^{\infty} c_k S(\chi_{E_k}) = \sum_{k=1}^{\infty} |S(\chi_{E_k})| = \sum_{k=1}^{\infty} |\nu(E_k)|$$

and since $S : L^p(X,\mu) \to \mathbb{R}$ then $\sum_{k=1}^{\infty} |\nu(E_k)| \in \mathbb{R}$, so $\sum_{k=1}^{\infty} \nu(E_k)$ converges absolutely and ν is in fact a signed measure.

Next, we claim that ν is absolutely continuous with respect to μ. If $E \in \mathcal{M}$ satisfies $\mu(E) = 0$ then χ_E is in the equivalence class containing the zero function, $[0] \in L^p(X,\mu)$. Since S is linear then it maps $0 \in L^p(X,\mu)$ to $0 \in \mathbb{R}$ and so $\nu(E) = S(\chi_E) = 0$, so that ν is, by definition (see Section 18.4, "The Radon-Nikodym Theorem") absolutely continuous with respect to μ.

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The Riesz Representation Theorem (continued 6)

Proof (continued). So by the Radon-Nikodym Theorem (actually, by Corollary 18.20, a corollary to the Radon-Nikodym Theorem) there is integrable function f such that

$$S(\chi_E) = \nu(E) = \int_E f\, f\mu \text{ for all } E \in \mathcal{M}.$$

Now each simple function $\varphi = \sum_{k=1}^n a_k \chi_{E_k}$ is in $L^p(X,\mu)$ (since each characteristic function on a measurable set in $L^P(X,\mu)$), then

$$S(\varphi) = S\left(\sum_{k=1}^n a_k \chi_{E_k}\right) = \sum_{k=1}^n a_k S(\chi_{E_k}) = \sum_{k=1}^n a_k \nu(E_k)$$

$$= \sum_{k=1}^n a_k \left(\int_{E_k} f\, d\mu\right) = \sum_{k=1}^n \left(\int_E f a_k \chi_{E_k}\, d\mu\right) = \int_X f\varphi\, d\mu. \qquad (*)$$

Since S is a bounded linear functional on $L^p(X,\mu)$, $|S(g)| \le \|S\| \|g\|_p$ for each $g \in L^p(X,\mu)$.

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Since S is a bounded linear functional on $L^p(X, \mu)$, $|S(g)| \leq \|S\| \|g\|_p$ for each $g \in L^p(X, \mu)$.

The Riesz Representation Theorem (continued 7)

Proof (continued). Therefore $|\int_X f\varphi\, d\mu| = |S(\varphi)| \le \|S\|\|\varphi\|_p$ for each simple function φ. So by Lemma 19.6 (with $M = \|S\|$), we have $f \in L^q(X,\mu)$. Now the functional $T_f = \int_X fg\, d\mu$ is a bounded linear functional (see the first definition in this section) and so is continuous. So the functional $g \mapsto S(g) - T_f(g)$ for all $g \in L^p(X,\mu)$ is continuous. But, since $S(\varphi) = T_f(\varphi) = \int_X f\varphi\, d\mu$ by $(*)$, bounded linear functional $S - T_f$ vanishes on the linear space of simple functions. By Theorem 19.5 (since $\mu(E) < \infty$ in this case), the linear space of simple functions is dense in $L^p(X,\mu)$, so $S - T_f$ vanishes on all of $L^p(X,\mu)$ (since $S - T_f$ is continuous) and $S = T_f$. That is every element $S \in (L^p(X,\mu))^*$ is the image under T of some $f \in L^q(X,\mu)$ so that T is onto. As argued at the beginning of this section, Hölder's Inequality shows that T_f is an isometry.

The Riesz Representation Theorem (continued 7)

Proof (continued). Therefore $|\int_X f\varphi\, d\mu| = |S(\varphi)| \le \|S\|\|\varphi\|_p$ for each simple function φ. So by Lemma 19.6 (with $M = \|S\|$), we have $f \in L^q(X, \mu)$. Now the functional $T_f = \int_X fg\, d\mu$ is a bounded linear functional (see the first definition in this section) and so is continuous. So the functional $g \mapsto S(g) - T_f(g)$ for all $g \in L^p(X, \mu)$ is continuous. But, since $S(\varphi) = T_f(\varphi) = \int_X f\varphi\, d\mu$ by $(*)$, bounded linear functional $S - T_f$ vanishes on the linear space of simple functions. By Theorem 19.5 (since $\mu(E) < \infty$ in this case), the linear space of simple functions is dense in $L^p(X, \mu)$, so $S - T_f$ vanishes on all of $L^p(X, \mu)$ (since $S - T_f$ is continuous) and $S = T_f$. That is every element $S \in (L^p(X, \mu))^*$ is the image under T of some $f \in L^q(X, \mu)$ so that T is onto. As argued at the beginning of this section, Hölder's Inequality shows that T_f is an isometry.

Next,

$$T(aS_1 + bS_2) = T_{aS_1+bS_2} = \int_X (aS_1 + bS_2)\cdot d\mu = a\int_X S_1 \cdot d\mu + b\int_X S_2 \cdot d\mu$$

$$= aT_{S_1} + bT_{S_2} = aT(S_1) + bT(S_2), \ldots$$

The Riesz Representation Theorem (continued 7)

Proof (continued). Therefore $|\int_X f\varphi\,d\mu| = |S(\varphi)| \le \|S\|\|\varphi\|_p$ for each simple function φ. So by Lemma 19.6 (with $M = \|S\|$), we have $f \in L^q(X,\mu)$. Now the functional $T_f = \int_X fg\,d\mu$ is a bounded linear functional (see the first definition in this section) and so is continuous. So the functional $g \mapsto S(g) - T_f(g)$ for all $g \in L^p(X,\mu)$ is continuous. But, since $S(\varphi) = T_f(\varphi) = \int_X f\varphi\,d\mu$ by $(*)$, bounded linear functional $S - T_f$ vanishes on the linear space of simple functions. By Theorem 19.5 (since $\mu(E) < \infty$ in this case), the linear space of simple functions is dense in $L^p(X,\mu)$, so $S - T_f$ vanishes on all of $L^p(X,\mu)$ (since $S - T_f$ is continuous) and $S = T_f$. That is every element $S \in (L^p(X,\mu))^*$ is the image under T of some $f \in L^q(X,\mu)$ so that T is onto. As argued at the beginning of this section, Hölder's Inequality shows that T_f is an isometry. Next,

$$T(aS_1+bS_2) = T_{aS_1+bS_2} = \int_X (aS_1+bS_2)\cdot d\mu = a\int_X S_1\cdot d\mu + b\int_X S_2\cdot d\mu$$

$$= aT_{S_1} + bT_{S_2} = aT(S_1) + bT(S_2), \ldots$$

The Riesz Representation Theorem (continued 8)

Proof (continued). ...so that T is an isometric isomorphism between linear spaces $L^q(X,\mu)$ and $(L^p(X,\mu))^*$, in the case of X that σ-finite.

Now suppose X is σ-finite. Then X is a countable union of finite measure sets $Y_1, Y_2, \ldots$. Define $X_n = \cup_{k=1}^n Y_k$ so that $\{X_n\}$ is an ascending sequence of measurable sets of finite measure whose union is X. Fix $n \in \mathbb{N}$. Since each X_n is finite in measure, then by the case $\mu(E) < \infty$, we know that we can find $f_n \in L^1(X_n,\mu)$ for which $S(g) = \int_{X_n} f_n g\, d\mu$ for all $g \in L^p(X_n,\mu)$ and $\int_{X_n} |f_n|^q\, d\mu = \|f_n\|_q^q \le \|S\|^q$ by Lemma 19.6 with $M = \|S\|$. Now we extend f_n from X_n to X be defining $f_n(x) = 0$ for $x \in X \setminus X_n$. Then for each $g \in L^p(X,\mu)$ with $g = 0$ on $X \setminus X_n$ we have $S(g) = \int_{X_n} f_n g\, d\mu = \int_X f_n g$ and $\int_X |f_n|^q\, d\mu = \int_X |f_n|^q \chi_E\, d\mu = \int_E |f_n|^q\, d\mu \le \|S\|^q$ for all such f_n. Now f_n is unique up to a set of measure zero since if f_n an df_n' satisfy $S(g) = \int_X f_n g\, d\mu = \int_X f_n' g\, d\mu$ for all $g \in L^p(X,\mu)$ with $g = 0$ on $X \setminus X_n$ then $\int_X (f_n - f_n') g\, d\mu = 0$.

The Riesz Representation Theorem (continued 8)

Proof (continued). ...so that T is an isometric isomorphism between linear spaces $L^q(X,\mu)$ and $(L^p(X,\mu))^*$, in the case of X that σ-finite.

Now suppose X is σ-finite. Then X is a countable union of finite measure sets $Y_1, Y_2, \ldots$. Define $X_n = \cup_{k=1}^n Y_k$ so that $\{X_n\}$ is an ascending sequence of measurable sets of finite measure whose union is X. Fix $n \in \mathbb{N}$. Since each X_n is finite in measure, then by the case $\mu(E) < \infty$, we know that we can find $f_n \in L^1(X_n,\mu)$ for which $S(g) = \int_{X_n} f_n g\, d\mu$ for all $g \in L^p(X_n,\mu)$ and $\int_{X_n} |f_n|^q\, d\mu = \|f_n\|_q^q \le \|S\|^q$ by Lemma 19.6 with $M = \|S\|$. Now we extend f_n from X_n to X be defining $f_n(x) = 0$ for $x \in X \setminus X_n$. Then for each $g \in L^p(X,\mu)$ with $g = 0$ on $X \setminus X_n$ we have $S(g) = \int_{X_n} f_n g\, d\mu = \int_X f_n g$ and $\int_X |f_n|^q\, d\mu = \int_X |f_n|^q \chi_E\, d\mu = \int_E |f_n|^q\, d\mu \le \|S\|^q$ for all such f_n. Now f_n is unique up to a set of measure zero since if f_n an df_n' satisfy $S(g) = \int_X f_n g\, d\mu = \int_X f_n' g\, d\mu$ for all $g \in L^p(X,\mu)$ with $g = 0$ on $X \setminus X_n$ then $\int_X (f_n - f_n')g\, d\mu = 0$.

The Riesz Representation Theorem (continued 9)

Proof (continued). Since we can consider $Y_+\{x \in X_n \mid f_n(x) > f_n'(x)\}$, $Y_- = \{x \in X_n \mid f_n(x) < f_n'(x)\}$, $g_+ = \chi_{Y_+}$, $g_- = \chi_{Y_-}$, and we have $\int_X (f_n - f_n' \chi_{Y_+} \, d\mu = \int_{Y_+} (f_n' - f_n) \, d\mu = 0$ and $\int_X (f_n - f_n' \chi_{Y_-} \, d\mu = \int_{Y_-} (f_n' - f_n) \, d\mu = -\int_{Y_-} (f_n - f_n') \, d\mu = 0$ so that by Exercise 18.19, $f_n - f_n' = 0$ a.e. on Y_+ an d$f_n' - f_n = 0$ a.e. on Y_-; that is, $\mu(Y_+) = \mu(Y_-) = 0$ an d$f_n = f_n'$ a.e. on X_n (and on X when extended). So we must have f_{n+1} restricted to X_n equal to f_n on X_n. We define f on X pointwise for $x \in \cup_{n=1}^{\infty} X_n$ as $f(x) = f_n(x)$ if $x \in X_n$. Then f is well-defined since $x \in X_i \cap X_j$ implies $f_i(x) = f_j(x)$. The sequence $\{f_n\}$ converges pointwise a.e. to f on X, so $\{|f|^q\}$ converges pointwise a.e. to $|f|^q$.

The Riesz Representation Theorem (continued 9)

Proof (continued). Since we can consider $Y_+\{x \in X_n \mid f_n(x) > f_n'(x)\}$, $Y_- = \{x \in X_n \mid f_n(x) < f_n'(x)\}$, $g_+ = \chi_{Y_+}$, $g_- = \chi_{Y_-}$, and we have $\int_X (f_n - f_n'\chi_{Y_+}\, d\mu = \int_{Y_+} (f_n' - f_n)\, d\mu = 0$ and $\int_X (f_n - f_n'\chi_{Y_-}\, d\mu = \int_{Y_-} (f_n' - f_n)\, d\mu = -\int_{Y_-} (f_n - f_n')\, d\mu = 0$ so that by Exercise 18.19, $f_n - f_n' = 0$ a.e. on Y_+ an d$f_n' - f_n = 0$ a.e. on Y_-; that is, $\mu(Y_+) = \mu(Y_-) = 0$ an d$f_n = f_n'$ a.e. on X_n (and on X when extended). So we must have f_{n+1} restricted to X_n equal to f_n on X_n. We define f on X pointwise for $x \in \cup_{n=1}^\infty X_n$ as $f(x) = f_n(x)$ if $x \in X_n$. Then f is well-defined since $x \in X_i \cap X_j$ implies $f_i(x) = f_j(x)$. The sequence $\{f_n\}$ converges pointwise a.e. to f on X, so $\{|f|^q\}$ converges pointwise a.e. to $|f|^q$. So by Fatou's Lemma

$$\int_X |f|^q\, d\mu = \int_X \lim |f_n|^q\, d\mu \le \liminf \int_X |f_n|^q\, d\mu \le \|S\|^q.$$

So $f \in L^q(X,\mu)$. Let $g \in L^p(X,\mu)$. For each $n \in \mathbb{N}$, define $g_n = g$ on X_n and $g_0 = 0$ for $x \in X \setminus X_n$.

The Riesz Representation Theorem (continued 9)

Proof (continued). Since we can consider $Y_+\{x \in X_n \mid f_n(x) > f_n'(x)\}$, $Y_- = \{x \in X_n \mid f_n(x) < f_n'(x)\}$, $g_+ = \chi_{Y_+}$, $g_- = \chi_{Y_-}$, and we have $\int_X (f_n - f_n'\chi_{Y_+}\, d\mu = \int_{Y_+}(f_n' - f_n)\, d\mu = 0$ and $\int_X (f_n - f_n'\chi_{Y_-}\, d\mu = \int_{Y_-}(f_n' - f_n)\, d\mu = -\int_{Y_-}(f_n - f_n')\, d\mu = 0$ so that by Exercise 18.19, $f_n - f_n' = 0$ a.e. on Y_+ an d$f_n' - f_n = 0$ a.e. on Y_-; that is, $\mu(Y_+) = \mu(Y_-) = 0$ an d$f_n = f_n'$ a.e. on X_n (and on X when extended). So we must have f_{n+1} restricted to X_n equal to f_n on X_n. We define f on X pointwise for $x \in \cup_{n=1}^\infty X_n$ as $f(x) = f_n(x)$ if $x \in X_n$. Then f is well-defined since $x \in X_i \cap X_j$ implies $f_i(x) = f_j(x)$. The sequence $\{f_n\}$ converges pointwise a.e. to f on X, so $\{|f|^q\}$ converges pointwise a.e. to $|f|^q$. So by Fatou's Lemma

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So $f \in L^q(X,\mu)$. Let $g \in L^p(X,\mu)$. For each $n \in \mathbb{N}$, define $g_n = g$ on X_n and $g_0 = 0$ for $x \in X \setminus X_n$.

The Riesz Representation Theorem (continued 10)

Proof (continued). By Hölder's Inequality (Theorem 19.1(i)) $|fg|$ is integrable on X and $|fg_n| \le |fg|$ a.e. on X. So, by the Lebesgue Dominated Convergence Theorem (Section 18.3)

$$\lim_{n\to\infty} \int_X fg_n \, d\mu = \int_X \lim_{n\to\infty} (fg_n) \, d\mu = \int_X fg \, d\mu. \qquad (14)$$

On the other hand, $\{|g - g_n|^p\} \to 0$ pointwise a.e. on X and $|g_n - g|^p \le |g|^p$ a.e. on X (since $g_n = g$ on X_n and $g_n = 0$ for $x \in X \setminus X_n$) for all $n \in \mathbb{N}$. Again by the Lebesgue Dominated Convergence Theorem,

$$\lim_{n\to\infty} \|g_n - g\|_p^p = \lim_{n\to\infty} \int_X |g_n - g|^p \, d\mu = \int_X \lim_{n\to\infty} |g_n - g|^p \, d\mu = \int_X |g - g| \, d\mu =$$

and so $\{g_n\| \to g$ in $L^p(X,\mu)$.

The Riesz Representation Theorem (continued 10)

Proof (continued). By Hölder's Inequality (Theorem 19.1(i)) $|fg|$ is integrable on X and $|fg_n| \le |fg|$ a.e. on X. So, by the Lebesgue Dominated Convergence Theorem (Section 18.3)

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$$\lim_{n\to\infty} \|g_n - g\|_p^p = \lim_{n\to\infty} \int_X |g_n - g|^p \, d\mu = \int_X \lim_{n\to\infty} |g_n - g|^p \, d\mu = \int_X |g - g| \, d\mu$$

and so $\{g_n\| \to g$ in $L^p(X,\mu)$. Since bounded linear functional S is continuous on $L^p(X,\mu)$, then $\lim_{n\to\infty} S(g_n) = S\left(\lim_{n\to\infty} g_n\right) = S(g)$. However, for each $n \in \mathbb{N}$, $S(g_n) = \int_{X_n} f_n g_n \, d\mu = \int_X fg_n \, d\mu$, so by (14), $S(g) = \lim_{n\to\infty} S(g_n) = \lim_{n\to\infty} \int_X fg_n \, d\mu = \int_X fg \, d\mu$.

The Riesz Representation Theorem (continued 10)

Proof (continued). By Hölder's Inequality (Theorem 19.1(i)) $|fg|$ is integrable on X and $|fg_n| \leq |fg|$ a.e. on X. So, by the Lebesgue Dominated Convergence Theorem (Section 18.3)

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On the other hand, $\{|g - g_n|^p\} \to 0$ pointwise a.e. on X and $|g_n - g|^p \leq |g|^p$ a.e. on X (since $g_n = g$ on X_n and $g_n = 0$ for $x \in X \setminus X_n$) for all $n \in \mathbb{N}$. Again by the Lebesgue Dominated Convergence Theorem,

$$\lim_{n\to\infty} \|g_n - g\|_p^p = \lim_{n\to\infty} \int_X |g_n - g|^p \, d\mu = \int_X \lim_{n\to\infty} |g_n - g|^p \, d\mu = \int_X |g - g| \, d\mu =$$

and so $\{g_n\| \to g$ in $L^p(X, \mu)$. Since bounded linear functional S is continuous on $L^p(X, \mu)$, then $\lim_{n\to\infty} S(g_n) = S\left(\lim_{n\to\infty} g_n\right) = S(g)$. However, for each $n \in \mathbb{N}$, $S(g_n) = \int_{X_n} f_n g_n \, d\mu = \int_X fg_n \, d\mu$, so by (14), $S(g) = \lim_{n\to\infty} S(g_n) = \lim_{n\to\infty} \int_X fg_n \, d\mu = \int_X fg \, d\mu$.

The Riesz Representation Theorem (continued 11)

The Riesz Representation Theorem for the Dual of $L^p(X, \mu)$.
Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space, let $1 \leq p < \infty$, and let q be the conjugate of p. For $f \in L^q(X, \mu)$ define $T_f \in (L^p(X, \mu))^*$ as $T_f(g) = \int_X fg\, d\mu$. Then $T : L^q(X, \mu) \to (L^p(X, \mu))^*$, defined as $T(f) = T_f$, is an isometric isomorphism.

Proof (continued). Again, we have $S(g) = \int_X fg\, d\mu$. So, as explained at the end of the case where $\mu(E) < \infty$, we have that $T : L^q(X, \mu) \to (L^p(X, \mu))^*$ is an isometric isomorphism. □