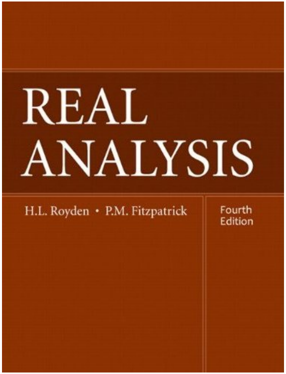
Real Analysis

Chapter 19. General L^p Spaces: Completeness, Duality, and Weak Convergence

19.3. The Kantorovitch Representation Theorem—Proofs of Theorems



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$$T_\nu(f) = \int_X f \, d\nu \text{ for all } r \in L^\infty(X, \mu).$$

Then $T : \mathcal{BFA}(X, \mathcal{M}, \mu) \to L^\infty(X, \mu)^*$, which maps ν to T_ν, is an isometric isomorphism of the normed linear space $\mathcal{BFA}(X, \mathcal{M}, \mu)$ onto the dual of $L^\infty(X, \mu)$.

Proof. We first show that T is an isometry. We saw above that for all $f \in L^\infty(X, \mu)$ and $\nu \in \mathcal{BFA}(X, \mathcal{M}, \mu)$ we have $|T_\nu(f)| = \left| \int_X f \, d\nu \right| \le \|\nu\|_{\text{var}} \|f\|_\infty$. Now the norm of a bounded linear functional T is

$$\|T\| = \inf \|M \mid |T(f)| \le M\|f\|_\infty \text{ for all } f\},$$

so we already have $\|T_\nu\| \le \|\nu\|_{\text{var}}$.

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so we already have $\|T_\nu\| \le \|\nu\|_{\text{var}}$.

Theorem 19.7 (continued 1)

Proof (continued). So we need to show $\|\nu\|_{\text{var}} \leq \|T\nu\|$. Let $\{E_k\}_{k=1}^n$ be a disjoint collection of sets in $\mathcal{M}$. For $1 \leq k \leq n$ define $c_k = \text{sgn}(\nu(E_k))$ and if $\varphi = \sum_{k=1}^n c_k \chi_{E_k}$. Then $\|\varphi\|_\infty = 1$ and

$$\sum_{k=1}^n |\nu(E_k)| = \int_X \varphi \, d\nu = T_\nu(\varphi) \leq \|T_\nu\| \|\varphi\|_\infty = \|T_\nu\|.$$

Therefore letting $\{E_k\}_{k=1}^n$ range over all finite disjoint collections in $\mathcal{M}$ and taking a supremum of the left hand side we get (since ν is a bounded finitely additive signed measure) $\|\nu\|_{\text{var}} \leq \|T_\nu\}$. Hence $\|T_\nu\| = \|\nu\|_{\text{var}}$ and so T is an isometry.

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In Exercise 19.3.B it is to be shown that $T : \mathcal{BFA}(X, \mathcal{M}, \mu) \to L^\infty(X, \mu)^*$ is linear (and so the homomorphism properties). If $\nu_1, \nu_2 \in \mathcal{BFA}(X, \mathcal{M}, \mu)$ with $\nu_1 \neq \nu_2$ then there is some $E \in \mathcal{M}$ where $\nu_1(E) \neq \nu_2(E)$.

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Theorem 19.7 (continued 2)

Proof (continued). Then with $f = \chi_E \in L^\infty(X,\mu)$ we have

$$T_{\nu_1}(f) = \int_X \chi_E \, d\nu_1 = \int_E 1 \, d\nu_1 = \nu_1(E)$$

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and so $T_{\nu_1} \neq T_{\nu_2}$. That is, $T(\nu_1) = T_{\nu_1} \neq T_{\nu_2} = T(\nu_2)$. So T is one to one. It remains to show that T is onto. Let $S \in L^\infty(X,\nu)^*$. Define $\nu : \mathcal{M} \to \mathbb{R}$ by $\nu(E) = S(\chi_E)$ for all $E \in \mathcal{M}$. Since $\chi_E \in L^\infty(X,\nu)$ then ν is "properly defined."

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$$\nu\left(\cup_{k=1}^n E_k\right) = S\left(\cup_{k=1}^n E_k\right) = S(\chi_{E_1} + \chi_{E_2} + \cdots + \chi_{E_n})$$

$$= S(\chi_{E_1}) + S(\chi_{E_2}) + \cdots + S(\chi_{E_n}) = \nu(E_1) + \nu(E_2) + \cdots + \nu(E_n),$$

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and so ν is finitely additive. If $E \in \mathcal{M}$ and $\mu(E) = 0$ then $\chi_E = 0$ a.e. on X and $\nu(E) = S(\chi_E) = S(0) = 0$. So ν is absolutely continuous with respect to μ.

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Proof (continued). Therefore $\nu \in \mathcal{BFA}(X, \mathcal{M}, \mu)$. For $\varphi = \sum_{k=1}^n c_k \chi_{E_k}$ (where $E_k = \{x \in X \mid \varphi(x) = c_k\}$) then

$$\int_X \varphi \, d\nu = \int_X \left(\sum k = 1^n c_k \chi_{E_k} \right) \, d\nu = \sum_{k=1}^n c_k \nu(E_k)$$

$$= \sum_{k=1}^n c_k S(E_k) = \sum_{k=1}^n S(c_k \chi_{E_k}) = S\left(\sum_{k=1}^n c_k \chi_{E_k} \right) = S(\varphi).$$

That is, $\int_X \varphi \, d\nu = S(\varphi)$ for all simple functions in $L^\infty(X, \mu)$. So $T_\nu = S$ on the set of all simple functions in $L^\infty(X, \mu)$. By the Simple Approximation Lemma (Section 18.1) the simple functions are dense in $L^\infty(X, \mu)$. So for any $f \in L^\infty(X, \mu)$ there is a sequence of simple functions $\{\varphi_n\}_{n=1}^\infty$ such that $\lim_{n \to \infty} \varphi_n = f$ (with respect to the $L^\infty(X, \mu)$ norm; that is, $\varphi_n \to f$ uniformly μ-a.e.). Let $g = f + 1$ on X. Since f is essentially bounded and $\nu(X)$ is finite, then g is essentially bounded and integrable on X with respect to ν.

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$$\lim_{n\to\infty} \int_X \varphi_n \, d\nu = \lim_{n\to\infty} S(\varphi_n) = S\left(\lim_{n\to\infty} \varphi_n\right) = S(f).$$

That is, $S(f) = \int_X f \, d\nu$. So arbitrary element S of $L^\infty(X,\mu)^*$ is the image under T of signed measure ν. That is, T is onto. So T is a isometric isomorphism, as claimed. □

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