Real Analysis

Chapter 19. General L^p Spaces: Completeness, Duality, and Weak Convergence

19.4. Weak Sequential Compactness in $L^p(X, \mu)$, $1 < p < \infty$—Proofs

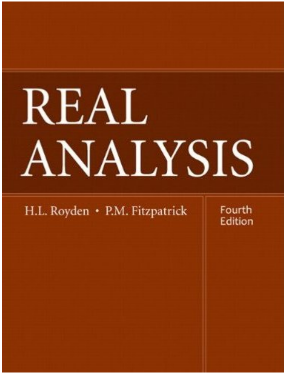

Table of contents

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$$(T_r(h))(g) = \int_X gh \, d\mu \text{ for all } h \in L^r \text{ and } g \in L^2$$

is an isometric isomorphism from L^r onto $(L^s)^*$.

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For reflexivity of L^p, let $S : (L^p)^* \to \mathbb{R}$ be an arbitrary (continuous) linear functional (all bounded linear functionals are continuous; consider the ε/δ definition of continuity for mappings between metric spaces and let $\delta = \varepsilon/\|T\|$ for nonzero linear functional T). We seek $f \in L^p$ for which $S = J(f)$ where $J : L^p \to (L^p)^{**}$ is the natural embedding (showing this will show that J is onto; here $S : L^p \to (L^p)^{**}$ also).

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Theorem 19.8 (continued 1)

Proof (continued). But for $T_q : L^q \to \mathbb{R}$ we have $S \circ T_q : L^q \to \mathbb{R}$ is the composition of continuous linear operators and so is a continuous linear functional on normed linear space L^q and so is bounded by Exercise 8.3. Therefore $S \circ T_q \in (L^q)^*$. By the proof of the Riesz Representation Theorem, mapping T_p defined above maps L^q *onto* $(L^q)^*$ and so there is $f \in L^q$ such that $T_p(f) = S \circ T_q$. So for all $g \in L^q$ we have $(S \circ T_q)(g) = T_p(f)(g)$ (and notice these are real numbers). Thus

$$\begin{aligned} S(T_q(g)) &= T_p(f)(g) = \int_X gf \text{ where } g \in L^q \text{ and } f \in L^p, \\ &\quad \text{by the definition of } T_p \\ &= \int_X fg = T_q(g)(f) \text{ by the definition of } T_q \\ &= J(f)(T_q(g)) \text{ since } f \in L^p \text{ and } g \in (L^p)^* = L^q \\ &\quad \text{(by the Riesz Representation Theorem.} \end{aligned}$$

Since T_q maps L^q onto $(L^p)^*$, then $S = J(f)$ on $(L^p)^*$.

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Proof (continued). Since S is an arbitrary mapping of $(L^p)^* \to \mathbb{R}$ (that is, S is an arbitrary element of L^{p**}), then $J : L^p \to L^{p**}$ is onto and $J(L^p) = L^{p**}$. That is (by definition), L^p is reflexive, as claimed. q □

The Riesz Weak Compactness Theorem

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Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space and $1 < p < \infty$. Then every bounded sequence in $L^p(X, \mu)$ has a weakly convergent subsequence; that is. if $\{f_n\}$ is a bounded sequence in $L^p(X, \mu)$, then there is a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ and a function $f \in L^p(X, \mu)$ for which

$$\lim_{k \to \infty} \int_X f_{n_k} g \, d\mu = \int_X fg \, d\mu \text{ for all } g \in L^q(X, \mu),$$

where $1/p + 1/q = 1$.

Proof. By Theorem 19.8, $L^p(X, \mu)$ is reflexive. By Theorem 14.17, every bounded sequence $\{f_n\}$ in a reflexive Banach space has a weakly convergent subsequence $\{f_{n_k}\}$ which converges, say, to $f \in L^p(X, \mu)$. That is, for every $\psi \in (L^p)^*$ we have $\lim_{n \to \infty} \psi(f_{n_k}) = \psi(f)$.

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The Riesz Weak Compactness Theorem (continued)

Proof (continued). But by the Reisz Representation Theorem, every $\psi \in (L^p)^*$ is of the form

$$\psi(f) = \int_X fg \, d\mu \text{ for all } f \in L^p(X, \mu) \text{ where } g \in L^q(X, \mu)$$

(and conversely each $g \in L^q(X, \mu)$ determines some bounded linear functional in $(L^p)^*$). So for all $g \in L^q(X, \mu)$, we have

$$\lim_{n \to \infty} \int_X f_{n_k} g \, d\mu = \int_X fg \, d\mu,$$

as claimed. □