Real Analysis

Chapter 19. General L^p Spaces: Completeness, Duality, and Weak Convergence

19.5. Weak Sequential Compactness in $L^1(X, \mu)$: The Dunford-Pettis Theorem—Proofs

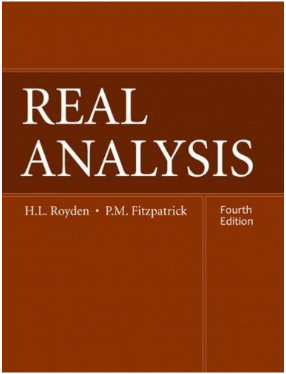

Table of contents

Proposition 19.10

Proposition 19.10. for a finite measure space $(X, \mathcal{M}, \mu)$ and bounded sequence $\{f_n\}$ in $L^1(X, \mu)$, the following two properties are equivalent:

(i) $\{f_n\}$ is uniformly integrable over X.

(ii) For each $\varepsilon > 0$, there is $M > 0$ such that

$$\int_{\{x \in X \mid |f_n(x)| \geq M\}} |f_n| \, d\mu < \varepsilon \text{ for all } n \in \mathbb{N}.$$

Proof. Since $\{f_n\}$ is bounded, there is $C > 0$ such that $\|f_n\|_1 \leq C$ for all $n \in \mathbb{N}$.

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Proof. Since $\{f_n\}$ is bounded, there is $C > 0$ such that $\|f_n\|_1 \leq C$ for all $n \in \mathbb{N}$.

Suppose $\{f_n\}$ is uniformly integrable over X. Let $\varepsilon > 0$. Then by the definition of "uniformly integrable," there is $\delta > 0$ such that if $E \subset X$ is measurable and

$$\text{if } \mu(E) < \delta \text{ then } \int_E |f_n| \, d\mu < \varepsilon \text{ for all } n \in \mathbb{N}.$$

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$$\text{if } \mu(E) < \delta \text{ then } \int_E |f_n| \, d\mu < \varepsilon \text{ for all } n \in \mathbb{N}.$$

Proposition 19.10 (continued 1)

Proof (continued). By Chebychev's Inequality (Section 18.2) for any positive $M > 0$

$$\mu(\{x \in X \mid |f_n(x)| \geq M\} \leq \frac{1}{M} \int_X |f_n| \, d\mu \leq \frac{C}{M}$$

for all $n \in \mathbb{N}$ So if $M > C/\delta$ (and so $C/M < \delta$) then $\mu(\{x \in X \mid |f_n(x)| \geq M\} < \delta$ and so by $(*)$, $\int_{\{x \in X \mid |f_n(x)| \geq M\}} |f_n| \, d\mu < \varepsilon$ for all $n \in \mathbb{N}$, and (i) implies (ii) as claimed.

Suppose (ii) holds. Let $\varepsilon > 0$. Choose $M > 0$ such that $\int_{\{x \in X \mid |f_n(x)| \geq M\}} |f_n| \, d\mu < \varepsilon/2$ for all $n \in \mathbb{N}$. Define $\delta = \varepsilon/(2M)$. Then if $E \subset X$ is measurable with $\mu(E) < \delta$ and is $n \in \mathbb{N}$ we have

$$\begin{aligned} \int_E |f_n| \, d\mu &= \int_{\{x \in X \mid |f_n(x)| \geq M\}} |f_n| \, d\mu + \int_{\{x \in X \mid |f_n(x)| < M\}} |f_n| \, d\mu \\ &< \frac{\varepsilon}{2} + M\mu(E) < \frac{\varepsilon}{2} + M\delta = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

since $\delta = \varepsilon/(2M)$ and so $M\delta = \varepsilon/2$.

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Proof (continued). By Chebychev's Inequality (Section 18.2) for any positive $M > 0$

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Suppose (ii) holds. Let $\varepsilon > 0$. Choose $M > 0$ such that $\int_{\{x \in X \mid |f_n(x)| \geq M\}} |f_n|\,d\mu < \varepsilon/2$ for all $n \in \mathbb{N}$. Define $\delta = \varepsilon/(2M)$. Then if $E \subset X$ is measurable with $\mu(E) < \delta$ and is $n \in \mathbb{N}$ we have

$$\int_E |f_n|\,d\mu = \int_{\{x \in X \mid |f_n(x)| \geq M\}} |f_n|\,d\mu + \int_{\{x \in X \mid |f_n(x)| < M\}} |f_n|\,d\mu$$
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Proposition 19.10 (continued 2)

Proposition 19.10. for a finite measure space $(X, \mathcal{M}, \mu)$ and bounded sequence $\{f_n\}$ in $L^1(X, \mu)$, the following two properties are equivalent:

(i) $\{f_n\}$ is uniformly integrable over X.

(ii) For each $\varepsilon > 0$, there is $M > 0$ such that

$$\int_{\{x \in X \mid |f_n(x)| \geq M\}} |f_n| \, d\mu < \varepsilon \text{ for all } n \in \mathbb{N}.$$

Proof (continued). Therefore $\{f_n\}$ is uniformly integrable over X and (i) holds, as claimed. □

Lemma 19.11

Lemma 19.11. For a finite measure space $(X, \mathcal{M}, \mu)$ and bounded uniformly integrable sequence $\{f_n\}$ in $L^1(X, \mu)$, there is a subsequence $\{f_{n_k}\}$ such that for each measurable subset E of X, the sequence of real numbers $\{\int_E f_{n_k}\, d\mu\}$ is Cauchy.

Proof. First, if $\{g_n\}$ is any bounded sequence in $L^1(X, \mu)$ and $\alpha > 0$, then the truncated sequence $\{g_n^{[\alpha]}\}$ is bounded in $L^2(X, \mu)$ since $\mu(X) < \infty$ (since then $g_n^{[\alpha]}$ is a bounded function on a set of finite measure for all $n \in \mathbb{N}$).

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Proof. First, if $\{g_n\}$ is any bounded sequence in $L^1(X, \mu)$ and $\alpha > 0$, then the truncated sequence $\{g_n^{[\alpha]}\}$ is bounded in $L^2(X, \mu)$ since $\mu(X) < \infty$ (since then $g_n^{[\alpha]}$ is a bounded function on a set of finite measure for all $n \in \mathbb{N}$). By the Riesz Compactness Theorem (Section 19.4; this requires $1 < p < \infty$, which is why we have moved to $L^2(X, \mu)$ where there is a subsequence $\{g_{n_k}^{[\alpha]}\}$ that converses weakly; that is, $\lim_{n\to\infty} T(g_{n_k}^{[\alpha]}) = T\left(\lim_{n\to\infty} g_{n_k}^{[\alpha]}\right)$ for all $T \in (L^2(X, \mu))^*$; see Section 8.2, "Weak Sequential Convergence in L^p").

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Lemma 19.11 (continued 1)

Proof (continued). Since $\mu(X) < \infty$, integration over a fixed measurable subset of X is a bounded linear functional on $L^2(X,\mu)$ (by Hölder's Inequality; see Section 19.2, "The Riesz Representation Theorem for the Dual of $L^p(X,\mu)$, $1 \le p < \infty$"; we take $g = 1$ to get integration over set E as the bounded linear functional on $L^2(X,\mu)$, $T(f) = \int_E fg\,d\mu = \int_E f\,d\mu$. Therefore for each measurable subset E of X, the sequence $\left\{\int_E g_{n_k}^{[\alpha]}\,d\mu\right\} = \{T(g_{n_k}^{[\alpha]})\}$ is a convergent sequence of real numbers and so is a Cauchy sequence of real numbers. We use this "observation" below.

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Now let $\{f_n\}$ be a bounded uniformly integrable sequence in $L^1(X, \mu)$. We use a diagonalization argument to find the desired subsequence $\{f_{n_k}\}$. By the "observation" above, there is a subsequence $\{f_n^1\}$ of $\{f_n\}$ which at the truncation level $\alpha = 1$ converges weakly in $L^2(X, \mu)$.

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Now let $\{f_n\}$ be a bounded uniformly integrable sequence in $L^1(X, \mu)$. We use a diagonalization argument to find the desired subsequence $\{f_{n_k}\}$. By the "observation" above, there is a subsequence $\{f_n^1\}$ of $\{f_n\}$ which at the truncation level $\alpha = 1$ converges weakly in $L^2(X, \mu)$.

Lemma 19.11 (continued 2)

Proof (continued). Since $\{f_n^1\}$ is also a bounded uniformly integrable sequence in $L^1(X,\mu)$, then by the "observation" there is a subsequence $\{f_n^2\}$ of $\{f_n^1\}$ which at the truncation level $\alpha = 2$ converges weakly in $L^2(X,\mu)$. We continue inductively to find a sequence of sequences, each of which is a subsequence of its predecessor and the kth subsequence $\{f_n^k\}$ at the truncation level $\alpha = k$ converges weakly in $L^2(X,\mu)$. Define the subsequence $\{h_n\}$ of $\{f_n\}$ as $j_n = f_n^n$ for $n \in \mathbb{N}$ ($\{h_n\}$ is the "diagonal sequence"). Then $\{h_n\}$ is a subsequence of $\{f_n\}$ and for each $k \in \mathbb{N}$ and

$$\text{for each measurable } E \subset X, \left\{ \int_E h_n^{[k]}\, d\mu \right\}_{n=1}^{\infty} \text{ is Cauchy,} \qquad (26)$$

by the "observation" above. Let $E \subset X$ be measurable. We claim that $\{\int_E h_n\, d\mu\}$ is a Cauchy sequence of real numbers.

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$$h_n - h_m = (h_n^{[k]} - h_m^{[k]}) + (h_m^{[k]} - h_m) + (h_n - h_n^{[k]}).$$

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Proof (continued). Since $\{f_n^1\}$ is also a bounded uniformly integrable sequence in $L^1(X,\mu)$, then by the "observation" there is a subsequence $\{f_n^2\}$ of $\{f_n^1\}$ which at the truncation level $\alpha = 2$ converges weakly in $L^2(X,\mu)$. We continue inductively to find a sequence of sequences, each of which is a subsequence of its predecessor and the kth subsequence $\{f_n^k\}$ at the truncation level $\alpha = k$ converges weakly in $L^2(X,\mu)$. Define the subsequence $\{h_n\}$ of $\{f_n\}$ as $j_n = f_n^n$ for $n \in \mathbb{N}$ ($\{h_n\}$ is the "diagonal sequence"). Then $\{h_n\}$ is a subsequence of $\{f_n\}$ and for each $k \in \mathbb{N}$ and

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Lemma 19.11 (continued 3)

Proof (continued). Therefore by (24) (in the Note before the statement of this lemma)

$$\left|\int_E (h_n - h_m)\,d\mu\right| \leq \left|\int_E (h_n^{[k]} - h_m^{[k]})\,d\mu\right| + \left|\int_E (h_m^{[k]} - h_m)\,d\mu\right|$$

$$+\left|\int_E (h_n - h_n^{[k]})\,d\mu\right| = \left|\int_E (h_n^{[k]} - h_m^{[k]})\,d\mu\right| + \int_{\{x\in E \mid |h_m(x)|>k\}} |h_m|\,d\mu$$

$$+\int_{\{x\in E \mid |h_n(x)|>k\}} |h_n|\,d\mu. \qquad (28)$$

Since $\{f_n\}$ is uniformly integrable by hypothesis (and so subsequence $\{h_n\}$ is uniformly integrable), then by Proposition 19.10 there is $k_0 \in \mathbb{N}$ such that

$$\int_{\{x\in E \mid |h_n(x)|>k_0\}} |h_n|\,d\mu < \varepsilon/3 \text{ for all } n \in \mathbb{N}. \qquad (29)$$

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$$\left|\int_E (h_n - h_m)\,d\mu\right| \le \left|\int_E (h_n^{[k]} - h_m^{[k]})\,d\mu\right| + \left|\int_E (h_m^{[k]} - h_m)\,d\mu\right|$$

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Lemma 19.11 (continued 4)

Proof (continued). From (26) with $k = k_0$, there is $N \in \mathbb{N}$ such that for all $n, m \geq \mathbb{N}$ we have

$$\left| \int_E h_n^{[k_0]} \, d\mu - \int_E h_m^{[k_0]} \, d\mu \right| = \left| \int_E (h_n^{[k_0]} - h_m^{[k_0]}) \, d\mu \right| < \varepsilon/3. \tag{30}$$

So for all $n, m \geq N$ we have from (28), (29) (applied to h_n and h_m), and (30) that

$$\left| \int_E (h_n - h_m) \, d\mu \right| = \left| \int_E h_n \, d\mu - \int_E h_m \, d\mu \right| < \varepsilon.$$

That is, sequence $\left\{ \int_E h_m \, d\mu \right\}_{n=1}^{\infty}$ is a Cauchy sequence of real numbers. So the claim holds where we take $\{f_{n_k}\}_{k=1}^{\infty} = \{h_k\}_{k=1}^{\infty}$. □

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Theorem 19.12. The Dunford-Pettis Theorem

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(ii) Every subsequence of $\{f_n\}$ has a further subsequence that converges weakly in $L^1(X, \mu)$.

Proof. Suppose $\{f_n\}$ is uniformly integrable. Since every subsequence of $\{f_n\}$ is bounded, to show (ii) it suffices to show that bounded sequence $\{f_n\}$ has a subsequence that converges weakly in $L^1(X, \mu)$. If we how that every nonnegative sequence $\{f_n\}$ has a weakly convergent subsequence, then the general result holds since we can apply this result to $\{f_n^+\}$ to find subsequence $\{f_{n_k}^+\}$ that converges weakly and then find a weakly convergent subsequence of $\{f_{n_k}^-\}$, say $\{f_{n_{k_\ell}}^-\}$. Then $\{f_{n_{k_\ell}}\}$ is a weakly convergent subsequence of $\{f_n\}$ since for any bounded linear function T we have. . .

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Theorem 19.12 (continued 1)

Proof (continued).

$$\begin{aligned}
\lim_{\ell\to\infty} T(f_{n_{k_\ell}}) &= \lim_{\ell\to\infty} T(f^+_{n_{k_\ell}} - f_{n_{k_\ell}}) \\
&= \lim_{\ell\to\infty} (T(f^+_{k_\ell}) - T(f^-_{n_{k_\ell}})) \text{ since } T \text{ is linear} \\
&= \lim_{\ell\to\infty} T(f^+_{k_\ell}) - \lim \ell \to \infty T(f^-_{k_\ell}) \\
&= T\left(\lim_{\ell\to\infty} f^+_{k_\ell}\right) - T\left(\lim_{\ell\to\infty} f^-_{k_\ell}\right) \text{ since } \{f^+_{k_\ell}\} \text{ and } \{f^-_{k_\ell}\} \\
&\quad \text{both converge weakly} \\
&= T\left(\lim_{\ell\to\infty} f^+_{k_\ell} - \lim_{\ell\to\infty} f^-_{k_\ell}\right) \text{ since } T \text{ is linear} \\
&= T\left(\lim_{\ell\to\infty} (f^+_{k_\ell} - f^-_{k_\ell})\right) = T\left(\lim_{\ell\to\infty} f_{k_\ell}\right).
\end{aligned}$$

So without loss of generality we may assume $\{f_n\}$ is nonnegative.

Theorem 19.12 (continued 2)

Proof (continued). By Lemma 19.11, there is a subsequence of $\{f_n\}$ which we denote $\{h_n\}$ such that for each measurable $E \subset X$ we have

$$\left\{ \int_E h_n \, d\mu \right\} \text{ is a Cauchy sequence of real numbers.} \tag{31}$$

For each $n \in \mathbb{N}$, define set function ν_n on $\mathcal{M}$ by $\nu_n(E) = \int_E h_n \, d\mu$ for all $E \in \mathcal{M}$. Since h_n is nonnegative then $\nu_n(E)$ is a measure absolutely continuous with respect to μ (see the first Note in Section 18.4, "The Radon-Nikodym Theorem"). Moreover, by (31) for each measurable $E \subset X$, $\{\nu_n(E)\}$ is a Cauchy sequence of real numbers. Define ν on $\mathcal{M}$ as $\nu(E) = \lim_{n\to\infty} \nu(E)$ for $E \in \mathcal{M}$ (so $\{\nu_n\}$ converges setwise to ν). Since $\{f_n\}$ is bounded in $L^1(X, \mu)$ by hypothesis then $\{h_n\} \subset \{f_n\}$ is bounded in $L^1(X, \mu)$ so from $\|h_n\|_1 = \int_X |h_n| \, d\mu = \int_X h_n \, d\mu = \nu_n(X)$ we see that $\{\nu_n(X)\}$ is a bounded sequence of real numbers. Therefore, the Vitali-Hahn-Saks Theorem (the "moreover" part) implies that ν is a measure on $(X, \mathcal{M})$ that is absolutely continuous with respect to μ.

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For each $n \in \mathbb{N}$, define set function ν_n on $\mathcal{M}$ by $\nu_n(E) = \int_E h_n \, d\mu$ for all $E \in \mathcal{M}$. Since h_n is nonnegative then $\nu_n(E)$ is a measure absolutely continuous with respect to μ (see the first Note in Section 18.4, "The Radon-Nikodym Theorem"). Moreover, by (31) for each measurable $E \subset X$, $\{\nu_n(E)\}$ is a Cauchy sequence of real numbers. Define ν on $\mathcal{M}$ as $\nu(E) = \lim_{n\to\infty} \nu(E)$ for $E \in \mathcal{M}$ (so $\{\nu_n\}$ converges setwise to ν). Since $\{f_n\}$ is bounded in $L^1(X,\mu)$ by hypothesis then $\{h_n\} \subset \{f_n\}$ is bounded in $L^1(X,\mu)$ so from $\|h_n\|_1 = \int_X |h_n| \, d\mu = \int_X h_n \, d\mu = \nu_n(X)$ we see that $\{\nu_n(X)\}$ is a bounded sequence of real numbers. Therefore, the Vitali-Hahn-Saks Theorem (the "moreover" part) implies that ν is a measure on $(X, \mathcal{M})$ that is absolutely continuous with respect to μ.

Theorem 19.12 (continued 3)

Proof (continued). By the Radon-Nikodym Theorem, there is nonnegative f on X such that $\nu(E) = \int_E f\,d\mu$ for all $E \in \mathcal{M}$ (since $\nu(E) < \infty$ then $f \in L^1(X,\mu)$). Since $\lim_{n\to\infty} \nu_n(E) = \lim_{n\to\infty} \int_E h_n\,d\mu = \int_E f\,d\mu = \nu(E)$ for all $E \in \mathcal{M}$, then for simple $\varphi = \sum_{k=1}^m c_k \chi_{E_k}$ we have

$$\lim_{n\to\infty}\left(\int_E h_n\varphi\,d\mu\right) = \lim_{n\to\infty}\left(\int_E h_n\left(\sum_{k=1}^m ck\chi_{E_k}\right)\,d\mu\right)$$

$$= \lim_{n\to\infty}\left(\sum_{k=1}^m c_k\int_E h_n\chi_{E_k}\,d\mu\right) = \lim_{n\to\infty}\sum_{k=1}^m c_k\left(\int_{E_k} h_n\,d\mu\right)$$

$$= \sum_{k=1}^m c_k \lim_{n\to\infty}\left(\int_{E_k} h_n\,d\mu\right) = \sum_{k=1}^m c_k\left(\int_{E_k} f\,d\mu\right) = \sum_{k=1}^m c_k\left(\int_E f\chi_{E_k}\,d\mu\right)$$

$$= \int_E f\left(\sum_{k=1}^m c_k\chi_{E_k}\right)\,d\mu = \int_E f\varphi\,d\mu. \qquad (32)$$

Theorem 19.12 (continued 4)

Proof (continued). Since all elements of $L^\infty(X,\mu)$ are essentially bounded, then be the Simple Approximation Lemma (Section 18.1) the set of simple functions is dense in $L^\infty(X,\mu)$. By hypothesis $\{f_n\}$ is bounded in $L^1(X,\mu)$ and so $\{h_n\} \subset \{f_n\}$ is bounded in $L^1(X,\mu)$, say by M. Let $\varepsilon > 0$. For any $g \in L^\infty(X,\mu)$ we have simple $\varphi \in L^\infty(X,\mu)$ with $\|\varphi - g\|_\infty < \min\{\varepsilon/(3M), \varepsilon/(3\|f\|_1\}$. From (32), there is $N \in \mathbb{N}$ such that for all $n \geq N$ $\left|\int_X h_n\varphi\,d\mu - \int_X f\varphi\,d\mu\right| < \varepsilon/3$. So for $n \geq N$

$$\left|\int_X h_n g\,d\mu - \int_X fg\,d\mu\right| = \left|\int_X h_n g\,d\mu - \int_X h_n\varphi\,d\mu + \int_X h_n\varphi\,d\mu\right.$$

$$\left. - \int_X f\varphi\,d\mu + \int_X f\varphi\,d\mu - \int_X fg\,d\mu\right| \leq \left|\int_X h_n g\,d\mu - \int_X h_n\varphi\,d\mu\right|$$

$$+ \left|\int_X h_n\varphi\,d\mu - \int_X f\varphi\,d\mu\right| + \left|\int_X f\varphi\,d\mu - \int_X fg\,d\mu\right| \ldots$$

Theorem 19.12 (continued 4)

Proof (continued). Since all elements of $L^\infty(X, \mu)$ are essentially bounded, then be the Simple Approximation Lemma (Section 18.1) the set of simple functions is dense in $L^\infty(X, \mu)$. By hypothesis $\{f_n\}$ is bounded in $L^1(X, \mu)$ and so $\{h_n\} \subset \{f_n\}$ is bounded in $L^1(X, \mu)$, say by M. Let $\varepsilon > 0$. For any $g \in L^\infty(X, \mu)$ we have simple $\varphi \in L^\infty(X, \mu)$ with $\|\varphi - g\|_\infty < \min\{\varepsilon/(3M), \varepsilon/(3\|f\|_1\}$. From (32), there is $N \in \mathbb{N}$ such that for all $n \geq N$ $\left|\int_X h_n\varphi\, d\mu - \int_X f\varphi\, d\mu\right| < \varepsilon/3$. So for $n \geq N$

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Theorem 19.12 (continued 5)

Proof (continued). ...

$$
\begin{aligned}
&= \left| \int_X h_n(g-\varphi)\,d\mu \right| + \left| \int_X h_n\varphi\,d\mu - \int_X f\varphi\,d\mu \right| + \left| \int_X f(\varphi-g)\,d\mu \right| \\
&\le \int_X |h_n||g-\varphi|\,d\mu + \left| \int_X h_n\varphi\,d\mu - \int_X f\varphi\,d\mu \right| + \int_X |f||\varphi-g|\,d\mu \\
&\le \|g-\varphi\|_\infty \int_X |h_n|\,d\mu + \left| \int_X h_n\varphi\,d\mu - \int_X f\varphi\,d\mu \right| + \|\varphi-g\|_\infty \int_X |f|\,d\mu \\
&= \|g-\varphi\|_\infty \|h_n\|_1 + \left| \int_X h_n\varphi\,d\mu - \int_X f\varphi\,d\mu \right| + \|\varphi-g\|_\infty \|f\|_1 \\
&< \frac{\varepsilon}{3M}M + \frac{\varepsilon}{3} + \frac{\varepsilon}{3\|f\|_1}\|f\|_1 = \varepsilon.
\end{aligned}
$$

Therefore

$$
\lim_{n\to\infty} \left(\int_X h_n g\,d\mu \right) = \int_X fg\,d\mu \text{ for all } g \in L^\infty(X,\mu). \qquad (33)
$$

Theorem 19.12 (continued 6)

Proof (continued). By the Riesz Representation Theorem (Section 19.2), every bounded linear functional on $L^1(X,\mu)$ is of the form $t_g = \int_X \cdot g\, d\mu$ for some $g \in L^\infty(X,\mu)$, so (33) implies $\lim n \to \infty T(h_n) = T(f)$ for all $T \in (L^1(X,\mu))^*$; that is, $\{h_n\}$ converges weakly in $L^1(X,\mu)$ to f, as claimed.

We now show that (ii) implies (i) by contradiction. ASSUME bounded $\{f_n\}$ satisfies (ii) but is not uniformly integrable. Then there is $\varepsilon_0 > 0$, a subsequence $\{h_n\}$ of $\{f_n\}$, and a sequence $\{E_n\}$ of measurable sets for which

$$\lim_{n\to\infty} \mu_n(E_n) = 0 \text{ but } \int_{E_n} |h_n|\, d\mu \geq \varepsilon_0 \text{ for all } n \in \mathbb{N} \qquad (34)$$

(see Exercise 19.5.A). Since we assume (ii) in this case, subsequence $\{h_n\}$ has a further subsequence that converges weakly in $L^1(X,\mu)$.

Theorem 19.12 (continued 6)

Proof (continued). By the Riesz Representation Theorem (Section 19.2), every bounded linear functional on $L^1(X,\mu)$ is of the form $t_g = \int_X \cdot g\, d\mu$ for some $g \in L^\infty(X,\mu)$, so (33) implies $\lim n \to \infty T(h_n) = T(f)$ for all $T \in (L^1(X,\mu))^*$; that is, $\{h_n\}$ converges weakly in $L^1(X,\mu)$ to f, as claimed.

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Theorem 19.12 (continued 7)

Proof (continued). But any subsequence of $\{h_n\}$ will also satisfy (34), so we can assume without loss of generality that $\{h_n\}$ itself converges weakly in $L^1(X, \mu)$ (or equivalently replace $\{h_n\}$ with the weakly convergent subsequence) to, say, h. For each $n \in \mathbb{N}$ define the signed measure ν_n on $\mathcal{M}$ by $\nu_n(E) = \int_E h_n \, d\mu$ for $E \in \mathcal{M}$ (if you like, we can first define the measures $\nu_n^+(E) = \int_E h_n^+ \, d\mu$ and $\nu_n^-(E) = \int_E h_n^- \, d\mu$ [see the introduction to Section 18.4, "The Radon-Nikodym Theorem"] and then define signed measure $\nu_n = \nu_n^+ - \nu_n^-$). Then each signed measure ν_n is absolutely continuous with respect to μ (since $\mu(E) = 0$ implies $\nu_n^+(E) = \nu_n^-(E) = \nu(E) = 0$). Since $\{h_n\}$ converges weakly in $L^1(X, \mu)$ to h then for $T_{\chi_E} = \int_E c\dot{\chi}_E \, d\mu$ a bounded linear functional on $L^1(X, \mu)$ (by Hölder's Inequality, since $\chi_E \in L^\infty(X, \mu)$; see Section 19.2) we have

$$\lim_{n\to\infty} T_{\chi_E}(h_n) = \lim_{n\to\infty} \int_E h_n \chi_E \, d\mu = \lim_{n\to\infty} \left(\int_E h_n \, d\mu \right)$$
$$= \lim_{n\to\infty} \nu_n(E) = T_{\chi_E}(h) = \int_E h \chi_E \, d\mu = \int_E h \, d\mu.$$

Theorem 19.12 (continued 7)

Proof (continued). But any subsequence of $\{h_n\}$ will also satisfy (34), so we can assume without loss of generality that $\{h_n\}$ itself converges weakly in $L^1(X,\mu)$ (or equivalently replace $\{h_n\}$ with the weakly convergent subsequence) to, say, h. For each $n \in \mathbb{N}$ define the signed measure ν_n on $\mathcal{M}$ by $\nu_n(E) = \int_E h_n \, d\mu$ for $E \in \mathcal{M}$ (if you like, we can first define the measures $\nu_n^+(E) = \int_E h_n^+ \, d\mu$ and $\nu_n^-(E) = \int_E h_n^- \, d\mu$ [see the introduction to Section 18.4, "The Radon-Nikodym Theorem"] and then define signed measure $\nu_n = \nu_n^+ - \nu_n^-$). Then each signed measure ν_n is absolutely continuous with respect to μ (since $\mu(E) = 0$ implies $\nu_n^+(E) = \nu_n^-(E) = \nu(E) = 0$). Since $\{h_n\}$ converges weakly in $L^1(X,\mu)$ to h then for $T_{\chi_E} = \int_E c\dot{\chi}_E \, d\mu$ a bounded linear functional on $L^1(X,\mu)$ (by Hölder's Inequality, since $\chi_E \in L^\infty(X,\mu)$; see Section 19.2) we have

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$$= \lim_{n\to\infty} \nu_n(E) = T_{\chi_E}(h) = \int_E h\chi_E \, d\mu = \int_E h \, d\mu.$$

Theorem 19.12 (continued 8)

Proof (continued). So sequence $\{\nu_n(E)\}$ of real numbers is convergent to $\int_E h_n \, d\mu$ and hence is a Cauchy sequence of real numbers for all $E \in \mathcal{M}$; that is, $\{\nu_n\}$ converges setwise to $\mathcal{M}$ to signed measure ν defines as $\nu(E) = \int_E h \, d\mu$ for $E \in \mathcal{M}$. So by the Vitali-Han-Saks Theorem extended to signed measures (see Exercise 19.22 in the 2019 "Updated Printing" of Royden and Fitzpatrick), sequence $\{\nu_n(E)\}$ is uniformly absolutely continuous with respect to μ (that is, for each $\varepsilon > 0$ there is $\delta > 0$ such that for any measurable $E \subset X$ and any $n \in \mathbb{N}$, if $\mu(E) < \delta$ then $\nu_n(E) < \varepsilon$). But this is a CONTRADICTION to (34). Therefore the assumption that $\{f_n\}$ is not uniformly integrable is false and, in fact, (i) holds, as claimed. □

Theorem 19.12 (continued 8)

Proof (continued). So sequence $\{\nu_n(E)\}$ of real numbers is convergent to $\int_E h_n \, d\mu$ and hence is a Cauchy sequence of real numbers for all $E \in \mathcal{M}$; that is, $\{\nu_n\}$ converges setwise to $\mathcal{M}$ to signed measure ν defines as $\nu(E) = \int_E h \, d\mu$ for $E \in \mathcal{M}$. So by the Vitali-Han-Saks Theorem extended to signed measures (see Exercise 19.22 in the 2019 "Updated Printing" of Royden and Fitzpatrick), sequence $\{\nu_n(E)\}$ is uniformly absolutely continuous with respect to μ (that is, for each $\varepsilon > 0$ there is $\delta > 0$ such that for any measurable $E \subset X$ and any $n \in \mathbb{N}$, if $\mu(E) < \delta$ then $\nu_n(E) < \varepsilon$). But this is a CONTRADICTION to (34). Therefore the assumption that $\{f_n\}$ is not uniformly integrable is false and, in fact, (i) holds, as claimed. □

Corollary 19.13

Corollary 19.13. Let $(X, \mathcal{M}, \mu)$ be a finite measure space and $\{f_n\}$ a sequence in $L^1(X, \mu)$ that is dominated by the function $g \in L^1(X, \mu)$ in the sense that

$$|f_n| \leq g \text{ a.e. on } E \text{ for all } n \in \mathbb{N}.$$

Then $\{f_n\}$ has a subsequence that converges weakly in $L^1(X, \mu)$.

Proof. The sequence $\{f_n\}$ is bounded in $L^1(X, \mu)$ since $\int_X |f_n| \, d\mu \leq \int_X |g| \, d\mu = \|g\|_1$. By Proposition 18.17, since g is integrable, for all $\varepsilon > 0$ there is $\delta > 0$ such that for any measurable $E \subset X$, if $\mu(E) < \delta$ then $\int_E |g| \, d\mu < \varepsilon$.

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