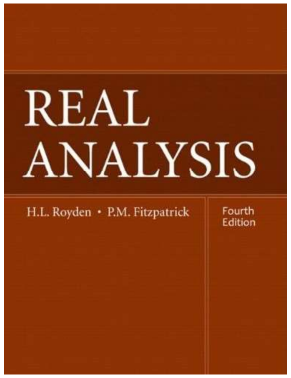
Real Analysis

Chapter 2. Lebesgue Measure

2.1. Introduction—Proofs of Theorems



Problem 2.1

Problem 2.1. Let m' be a set function defined on a σ-algebra $\mathcal{A}$ with values in $[0, \infty]$. Assume m' is countably additive over countable disjoint collections in $\mathcal{A}$. If A and B are two sets in $\mathcal{A}$ with $A \subset B$, then $m'(A) \le m'(B)$. This is called *monotonicity*.

Proof. First, $B \setminus A = B \cap A^c$ and since $\mathcal{A}$ is a σ-algebra (and hence closed under countable intersections and complements), then $B \setminus A \in \mathcal{A}$. Next, $B = (B \setminus A) \,\dot{\cup}\, A$, so by the hypothesized Countable Additivity, $m'(B) = m'(B \setminus A) + m'(A)$ since $B \setminus A$ and A are disjoint. Since $m'(B \setminus A) \ge 0$ by hypothesis, then $m'(A) \le m'(B)$. □

Note. We could weaken the hypothesis of "σ-algebra" to "algebra" and weaken the hypothesis of "countable additivity" to "finite additivity," and the result would still hold.