Real Analysis

Chapter 2. Lebesgue Measure

2.1. Introduction—Proofs of Theorems

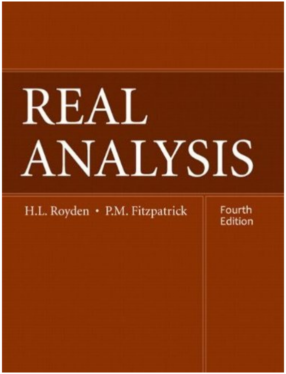

Table of contents

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Proof. First, $B \setminus A = B \cap A^c$ and since $\mathcal{A}$ is a σ-algebra (and hence closed under countable intersections and complements), then $B \setminus A \in \mathcal{A}$. Next, $B = (B \setminus A) \cup A$, so by the hypothesized Countable Additivity, $m'(B) = m'(B \setminus A) + m'(A)$ since $B \setminus A$ and A are disjoint. Since $m'(B \setminus A) \geq 0$ by hypothesis, then $m'(A) \leq m'(B)$. □

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