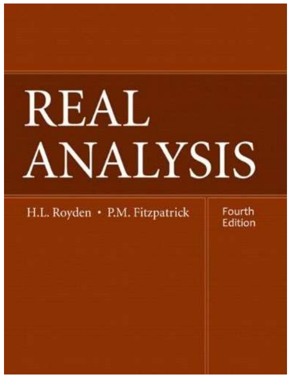
Real Analysis

Chapter 2. Lebesgue Measure

2.2. Lebesgue Outer Measure—Proofs of Theorems



Lemma 2.2.A

Lemma 2.2.A. Outer measure is monotone. That is, if $A \subset B$ then $m^*(A) \le m^*(B)$.

Proof. Let $A \subset B$ be sets of real numbers. We consider the sets

$$X_A = \left\{ \sum_{n=1}^{\infty} \ell(I_n) \,\middle|\, A \subset \cup_{n=1}^{\infty} I_n \text{ and each } I_n \text{ is a bounded open interval} \right\}$$

and

$$X_B = \left\{ \sum_{n=1}^{\infty} \ell(I_n) \,\middle|\, B \subset \cup_{n=1}^{\infty} I_n \text{ and each } I_n \text{ is a bounded open interval} \right\}.$$

To find an arbitrary element of X_B, we need an arbitrary countable covering of B by bounded open intervals.

Lemma 2.2.A (continued)

Lemma 2.2.A. Outer measure is monotone. That is, if $A \subset B$ then $m^*(A) \le m^*(B)$.

Proof (continued). To find an arbitrary element of X_B, we need an arbitrary countable covering of B by bounded open intervals. So let $\{I_n\}_{n=1}^{\infty}$ be a countable collection of bounded open intervals such that $B \subset \cup_{n=1}^{\infty} I_n$. Then $\sum_{n=1}^{\infty} \ell(I_n) \in X_B$. Notice that $A \subset B \subset \cup_{n=1}^{\infty} I_n$ and hence $\sum_{n=1}^{\infty} \ell(I_n) \in X_A$. So $X_B \subset X_A$. Therefore

$$m^*(A) = \inf(X_A) \le \inf(X_B) = m^*(B).$$

□

Proposition 2.1

Proposition 2.1. The outer measure of an interval is its length.

Proof. (1) We first show the result holds for a closed interval $[a, b]$. Let $\varepsilon > 0$. Then $(a - \varepsilon/2, b + \varepsilon/2)$ is (alone) a covering of $[a, b]$ and $\ell((a - \varepsilon/2, b + \varepsilon/2)) = b - a + \varepsilon$. Since ε is arbitrary, $m^*([a, b]) \le b - a = \ell([a, b])$.

Next, let $\{I_n\}$ be a covering of $[a, b]$ by bounded open intervals. By the Heine-Borel Theorem, there exists a finite subset A of I_n's covering $[a, b]$. So $a \in I_1$ for some $I_1 = (a_1, b_1) \in A$. Also, if $b_1 \le b$, then $b_1 \in I_2$ for some $I_2 = (a_2, b_2) \in A$. Similarly, we can construct $I_1, I_2, \ldots, I_k$ (say, $(a_1, b_1), (a_2, b_2), \ldots, (a_k, b_k)$ such that $a_i < b_{i-1} < b_i$). Then

$$\begin{aligned} \sum \ell(I_n) &\ge \sum_{i=1}^{k} \ell(I_i) = \sum_{i=1}^{k} (b_i - a_i) \\ &= (b_k - a_k) + (b_{k-1} - a_{k-1}) + \cdots + (b_1 - a_1) \\ &= b_k - (a_k - b_{k-1}) - \cdots - (a_2 - b_1) - a_1 \\ &> b_k - a_1. \end{aligned}$$

Proposition 2.1 (continued)

$$\sum \ell(I_n) > b_k - a_1.$$

Since $a_1 < a$ and $b_k > b$, then $\sum \ell(I_n) > b - a$. So $m^*([a,b]) = b - a = \ell([a,b])$.

(2) Next, consider an arbitrary bounded interval I. Then for any $\varepsilon > 0$, there is a closed interval $J \subset I$ such that $\ell(J) > \ell(I) - \varepsilon$. Notice that $m^*(I) \le m^*(\bar{I})$ by monotonicity. So

$$\begin{aligned}
\ell(I) - \varepsilon < \ell(J) &= m^*(J) \text{ by (1), since } J \text{ is a closed bounded interval} \\
&\le m^*(I) \text{ by monotonicity (Lemma 2.2.A)} \\
&\le m^*(\bar{I}) \text{ by monotonicity since } I \subseteq \bar{I} \\
&= \ell(\bar{I}) \text{ by (1), since } I \text{ is a closed bounded interval} \\
&= \ell(I) \text{ since } I \text{ is a bounded interval}
\end{aligned}$$

and therefore $\ell(I) - \varepsilon < m^*(I) \le \ell(I)$. Since ε is arbitrary, $\ell(I) = m^*(I)$.

Proposition 2.1 (continued 2)

Proposition 2.1. The outer measure of an interval is its length.

Proof (continued). (3) If I is an unbounded interval, then given any natural number $n \in \mathbb{N}$, there is a closed interval $J \subset I$ with $\ell(J) = n$. Hence $m^*(I) \ge m^*(J) = \ell(J) = n$. Since $m^*(I) \ge n$ and $n \in \mathbb{N}$ is arbitrary, then $m^*(I) = \infty = \ell(I)$. □

Proposition 2.2

Proposition 2.2. Outer measure is translation invariant; that is, for any set A and number y, $m^*(A + y) = m^*(A)$.

Proof. Suppose $m^*(A) = M < \infty$. Then for all $\varepsilon > 0$ there exist $\{I_n\}_{n=1}^{\infty}$ bounded open intervals such that $A \subset \cup I_n$ and $\sum \ell(I_n) < M + \varepsilon$ by Theorem 0.3(b). So if $y \in \mathbb{R}$, then $\{I_n + y\}$ is a covering of $A + y$ and so $m^*(A + y) \le \sum \ell(I_n + y) = \sum \ell(I_n) < M + \varepsilon$. Therefore $m^*(A + y) \le M$.

Now let $\{J_n\}$ be a collection of bounded open intervals such that $\cup J_n \supset A + y$. ASSUME that $\sum \ell(J_n) < M$. Then $\{J_n - y\}$ is a covering of A and $\sum \ell(J_n - y) = \sum \ell(J_n) < M$, a CONTRADICTION. So $\sum \ell(J_n) \ge M$ and hence $m^*(A + y) \ge M$. So $m^*(A + y) = m^*(A) = M$.

Proposition 2.2 (continued)

Proposition 2.2. Outer measure is translation invariant; that is, for any set A and number y, $m^*(A + y) = m^*(A)$.

Proof (continued). Suppose $m^*(A) = \infty$. Then for any $\{I_n\}_{n=1}^{\infty}$ a set of bounded open intervals such that $A \subset \cup I_n$, we must have $\sum \ell(I_n) = \infty$. Consider $A + y$. For any $\{J_n\}_{n=1}^{\infty}$ a set of bounded open intervals such that $A + y \subset \cup J_n$, the collection $\{J_n - y\}_{n=1}^{\infty}$ is a set of bounded open intervals such that $A \subset \cup(J_n - y)$. So $\sum \ell(J_n - y) = \infty$. But $\ell(J_n) = \ell(J_n - y)$, so we must have $\sum \ell(J_n) = \infty$. Since $\{J_n\}_{n=1}^{\infty}$ is an arbitrary collection of bounded open intervals covering $A + y$, we must have $m^*(A + y) = \infty = m^*(A)$. □

Proposition 2.3

Proposition 2.3. Outer measure is countably subadditive. That is, if $\{E_k\}_{k=1}^{\infty}$ is any countable collection of sets then

$$m^*\left(\bigcup_{k=1}^{\infty} E_k\right) \le \sum_{k=1}^{\infty} m^*(E_k).$$

Proof. The result holds trivially if $m^*(E_k) = \infty$ for some k. So without loss of generality assume each E_k has finite outer measure. Then for all $\varepsilon > 0$ and for each $k \in \mathbb{N}$, there is a countable set of open intervals $\{I_{k,i}\}_{i=1}^{\infty}$ such that $E_k \subset \cup_{i=1}^{\infty} I_{k,i}$ and $\sum_{i=1}^{\infty} \ell(I_{k,i}) < m^*(E_k) + \varepsilon/2^k$ (by Theorem 0.3(b)). Then $\{I_{k,i}\}$ where $i, k \in \mathbb{N}$ is a countable collection (by Theorem 0.10) of open intervals that covers $\cup E_k$. So $m^*(\cup E_k) \le \sum_{i,k} \ell(I_{k,i}) = \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} \ell(I_{k,i}) < \sum_{k=1}^{\infty}(m^*(E_k) + \varepsilon/2^k) = (\sum_{k=1}^{\infty} m^*(E_k)) + \varepsilon$.

Therefore $m^*\left(\bigcup_{k=1}^{\infty} E_k\right) \le \sum_{k=1}^{\infty} m^*(E_k)$. □

Exercise 2.5

Exercise 2.5. $[0, 1]$ is not countable.

Proof. By Proposition 2.1, $m^*([0, 1]) = \ell([0, 1]) = 1$. By Corollary 6-9 (in Kirkwood's book and in the Riemann-Lebesgue Supplement; or by the Example on page 31 of Royden and Fitzpatrick), if a set is countable then the outer measure is 0, or by the logically equivalent contrapositive, if a set has positive measure then it is not countable. Hence $[0, 1]$ is not countable. □