Proposition 2.1

The outer measure of an interval is its length.

Proof. (1) We first show the result holds for a closed interval $[a, b]$. Let $\varepsilon > 0$. Then $(a - \varepsilon/2, b + \varepsilon/2)$ is (alone) a covering of $[a, b]$ and
\[
\ell((a - \varepsilon/2, b + \varepsilon/2)) = b - a + \varepsilon.
\]
Since $\varepsilon$ is arbitrary,
\[
m^*([a, b]) = b - a = \ell([a, b]).
\]
Next, let \{l_i\} be a covering of $[a, b]$ by bounded open intervals. By the Heine-Borel theorem, there exists a finite subset $A$ of $l_n$'s covering $[a, b]$. So $a \in l_1$ for some $l_1 = (a_1, b_1) \in A$. Also, if $h_1 \leq b$, then $h_1 \in l_2$ for some $l_2 = (a_2, b_2) \in A$. Similarly, we can construct $l_1, l_2, \ldots, l_k$ (say, $(a_1, b_1), (a_2, b_2), \ldots, (a_k, b_k)$ such that $a_i < b_{i-1} < b_i$). Then
\[
\sum_i \ell(l_i) \geq \sum_i \ell(l_i) = \sum_{i=1}^k(b_i - a_i) = (b_k - a_k) + (b_{k-1} - a_{k-1}) + \cdots + (b_1 - a_1) = b_k - a_k - (a_{k-1} - b_{k-1}) - \cdots - (a_2 - b_1) - a_1 > b_k - a_1.
\]

(2) Next, consider an arbitrary bounded interval $I$. Then for any $\varepsilon > 0$, there is a closed interval $J \subset I$ such that $\ell(J) > \ell(I) - \varepsilon$. Notice that $m^*(I) \leq m^*(J)$ by monotonicity. So
\[
\ell(I) - \varepsilon < \ell(J) = m^*(J) \leq m^*(I),
\]
\[
\ell(I) - \varepsilon < m^*(I) \leq \ell(I).
\]

Proposition 2.1 (continued)

Proposition 2.1 (continued 2)

Proposition 2.1. The outer measure of an interval is its length.

Proof (continued). (3) If $I$ is an unbounded interval, then given any natural number $n \in \mathbb{N}$, there is a closed interval $J \subset I$ with $\ell(J) = n$. Hence $m^*(I) \geq m^*(J) = \ell(J) = n$. Since $m^*(I) \geq n$ and $n \in \mathbb{N}$ is arbitrary, then $m^*(I) = \infty = \ell(I)$.
Proposition 2.2. Outer measure is translation invariant; that is, for any set $A$ and number $y$, $m^*(A + y) = m^*(A)$.

**Proof.** Suppose $m^*(A) = M < \infty$. Then for all $\varepsilon > 0$ there exist $\{I_n\}_{n=1}^{\infty}$ bounded open intervals such that $A \subset \bigcup I_n$ and $\sum \ell(I_n) < M + \varepsilon$ by Theorem 0.3(b). So if $y \in \mathbb{R}$, then $\{I_n + y\}$ is a covering of $A + y$ and so $m^*(A + y) \leq \sum \ell(I_n + y) = \sum \ell(I_n) < M + \varepsilon$. Therefore $m^*(A + y) \leq M$.

Now let $\{J_n\}$ be a collection of bounded open intervals such that $\bigcup J_n \supseteq A + y$. ASSUME that $\sum \ell(J_n) < M$. Then $\{J_n - y\}$ is a covering of $A$ and $\sum \ell(J_n - y) = \sum \ell(J_n) < M$, a CONTRADICTION. So $\sum \ell(J_n) \geq M$ and hence $m^*(A + y) \geq M$. So $m^*(A + y) = m^*(A) = M$.

---

Proposition 2.3. Outer measure is countably subadditive. That is, if $\{E_k\}_{k=1}^{\infty}$ is any countable collection of sets then

$$m^* \left( \bigcup_{k=1}^{\infty} F_k \right) \leq \sum_{k=1}^{\infty} m^*(F_k)$$

**Proof.** The result holds trivially if $m^*(E_k) = \infty$ for some $k$. So without loss of generality assume each $E_k$ has finite outer measure. Then for all $\varepsilon > 0$ and for each $k \in \mathbb{N}$, there is a countable set of open intervals $\{I_{k,i}\}_{i=1}^{\infty}$ such that $E_k \subset \bigcup_{i=1}^{\infty} I_{k,i}$ and $\sum_{i=1}^{\infty} \ell(I_{k,i}) < m^*(E_k) + \varepsilon / 2^k$ (by Theorem 0.3(b)). Then $\{I_{k,i}\}$ where $i, k \in \mathbb{N}$ is a countable collection (by Theorem 0.10) of open intervals that covers $\bigcup E_k$. So $m^*(\bigcup E_k) \leq \sum_{k=1}^{\infty} \ell(I_{k,i}) = \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} \ell(I_{k,i}) < \sum_{k=1}^{\infty} (m^*(E_k) + \varepsilon / 2^k) = (\sum_{k=1}^{\infty} m^*(E_k)) + \varepsilon$.

Therefore $m^*(\bigcup_{k=1}^{\infty} E_k) \leq \sum_{k=1}^{\infty} m^*(E_k)$.

---