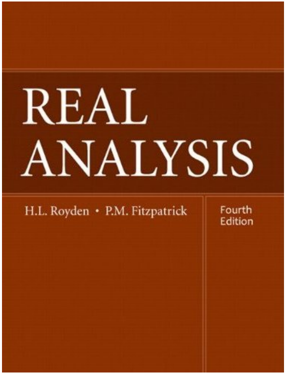
Real Analysis

Chapter 2. Lebesgue Measure

2.2. Lebesgue Outer Measure—Proofs of Theorems



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Lemma 2.2.A

Lemma 2.2.A. Outer measure is monotone. That is, if $A \subset B$ then $m^*(A) \leq m^*(B)$.

Proof. Let $A \subset B$ be sets of real numbers. We consider the sets

$$X_A = \left\{ \sum_{n=1}^{\infty} \ell(I_n) \,\middle|\, A \subset \cup_{n=1}^{\infty} I_n \text{ and each } I_n \text{ is a bounded open interval} \right\}$$

and

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$$m^*(A) = \inf(X_A) \leq \inf(X_B) = m^*(B).$$

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Next, let $\{I_n\}$ be a covering of $[a, b]$ by bounded open intervals. By the Heine-Borel Theorem, there exists a finite subset A of I_n's covering $[a, b]$. So $a \in I_1$ for some $I_1 = (a_1, b_1) \in A$. Also, if $b_1 \leq b$, then $b_1 \in I_2$ for some $I_2 = (a_2, b_2) \in A$. Similarly, we can construct $I_1, I_2, \ldots, I_k$ (say, $(a_1, b_1), (a_2, b_2), \ldots, (a_k, b_k)$ such that $a_i < b_{i-1} < b_i$).

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$$\begin{aligned} \sum \ell(I_n) &\geq \sum_{i=1}^{k} \ell(I_i) = \sum_{i=1}^{k} (b_i - a_i) \\ &= (b_k - a_k) + (b_{k-1} - a_{k-1}) + \cdots + (b_1 - a_1) \\ &= b_k - (a_k - b_{k-1}) - \cdots - (a_2 - b_1) - a_1 \\ &> b_k - a_1. \end{aligned}$$

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Proposition 2.1 (continued)

$$\sum \ell(I_n) > b_k - a_1.$$

Since $a_1 < a$ and $b_k > b$, then $\sum \ell(I_n) > b - a$. So $m^*([a, b]) = b - a = \ell([a, b])$.

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(2) Next, consider an arbitrary bounded interval I. Then for any $\varepsilon > 0$, there is a closed interval $J \subset I$ such that $\ell(J) > \ell(I) - \varepsilon$. Notice that $m^*(I) \le m^*(\bar{I})$ by monotonicity. So

$$\begin{aligned}
\ell(I) - \varepsilon < \ell(J) &= m^*(J) \text{ by (1), since } J \text{ is a closed bounded interval} \\
&\le m^*(I) \text{ by monotonicity (Lemma 2.2.A)} \\
&\le m^*(\bar{I}) \text{ by monotonicity since } I \subseteq \bar{I} \\
&= \ell(\bar{I}) \text{ by (1), since } I \text{ is a closed bounded interval} \\
&= \ell(I) \text{ since } I \text{ is a bounded interval}
\end{aligned}$$

and therefore $\ell(I) - \varepsilon < m^*(I) \le \ell(I)$.

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$$\begin{array}{rcll} \ell(I) - \varepsilon < \ell(J) & = & m^*(J) & \text{by (1), since } J \text{ is a closed bounded interval} \\ & \leq & m^*(I) & \text{by monotonicity (Lemma 2.2.A)} \\ & \leq & m^*(\bar{I}) & \text{by monotonicity since } I \subseteq \bar{I} \\ & = & \ell(\bar{I}) & \text{by (1), since } I \text{ is a closed bounded interval} \\ & = & \ell(I) & \text{since } I \text{ is a bounded interval} \end{array}$$

and therefore $\ell(I) - \varepsilon < m^*(I) \leq \ell(I)$. Since ε is arbitrary, $\ell(I) = m^*(I)$.

Proposition 2.1 (continued 2)

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Proof (continued). (3) If I is an unbounded interval, then given any natural number $n \in \mathbb{N}$, there is a closed interval $J \subset I$ with $\ell(J) = n$. Hence $m^*(I) \geq m^*(J) = \ell(J) = n$. Since $m^*(I) \geq n$ and $n \in \mathbb{N}$ is arbitrary, then $m^*(I) = \infty = \ell(I)$. □

Proposition 2.2

Proposition 2.2. Outer measure is translation invariant; that is, for any set A and number y, $m^*(A+y) = m^*(A)$.

Proof. Suppose $m^*(A) = M < \infty$. Then for all $\varepsilon > 0$ there exist $\{I_n\}_{n=1}^{\infty}$ bounded open intervals such that $A \subset \cup I_n$ and $\sum \ell(I_n) < M + \varepsilon$ by Theorem 0.3(b).

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Now let $\{J_n\}$ be a collection of bounded open intervals such that $\cup J_n \supset A + y$. ASSUME that $\sum \ell(J_n) < M$. Then $\{J_n - y\}$ is a covering of A and $\sum \ell(J_n - y) = \sum \ell(J_n) < M$, a CONTRADICTION. So $\sum \ell(J_n) \geq M$ and hence $m^*(A+y) \geq M$. So $m^*(A+y) = m^*(A) = M$.

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$$m^*\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} m^*(E_k).$$

Proof. The result holds trivially if $m^*(E_k) = \infty$ for some k. So without loss of generality assume each E_k has finite outer measure.

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Therefore $m^*\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} m^*(E_k)$. □

Exercise 2.5

Exercise 2.5. $[0,1]$ is not countable.

Proof. By Proposition 2.1, $m^*([0,1]) = \ell([0,1]) = 1$. By Corollary 6-9 (in Kirkwood's book and in the Riemann-Lebesgue Supplement; or by the Example on page 31 of Royden and Fitzpatrick), if a set is countable then the outer measure is 0, or by the logically equivalent contrapositive, if a set has positive measure then it is not countable. Hence $[0,1]$ is not countable. □

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