Chapter 2. Lebesgue Measure

2.2. Lebesgue Outer Measure—Proofs of Theorems
Proposition 2.1

Proposition 2.1. The outer measure of an interval is its length.

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Next, let $\{I_n\}$ be a covering of $[a, b]$ by bounded open intervals. By the Heine-Borel Theorem, there exists a finite subset $A$ of $I_n$’s covering $[a, b]$. 
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Next, let $\{I_n\}$ be a covering of $[a, b]$ by bounded open intervals. By the Heine-Borel Theorem, there exists a finite subset $A$ of $I_n$'s covering $[a, b]$. So $a \in I_1$ for some $I_1 = (a_1, b_1) \in A$. Also, if $b_1 \leq b$, then $b_1 \in I_2$ for some $I_2 = (a_2, b_2) \in A$. Similarly, we can construct $I_1, I_2, \ldots, I_k$ (say, $(a_1, b_1), (a_2, b_2), \ldots, (a_k, b_k)$ such that $a_i < b_{i-1} < b_i$).
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\[
\sum_{i=1}^{k} \ell(I_i) \geq \sum_{i=1}^{k} (b_i - a_i) = b_k - a_k - (a_k - b_{k-1}) - \cdots - (a_2 - b_1) - a_1 > b_k - a_1.
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\[
\sum_{i=1}^{k} \ell(I_i) = \sum_{i=1}^{k} (b_i - a_i) = b_k - a_1.
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Proposition 2.1 (continued)

\[ \sum \ell(I_n) > b_k - a_1. \]

Since \( a_1 < a \) and \( b_k > b \), then \( \sum \ell(I_n) > b - a \). So \( m^*([a, b]) = b - a = \ell([a, b]) \).

(2) Next, consider an arbitrary bounded interval \( I \).
Proposition 2.1 (continued)

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Since \( a_1 < a \) and \( b_k > b \), then \( \sum \ell(I_n) > b - a \). So \( m^*([a, b]) = b - a = \ell([a, b]) \).

(2) Next, consider an arbitrary bounded interval \( I \). Then for any \( \varepsilon > 0 \), there is a closed interval \( J \subset I \) such that \( \ell(J) > \ell(I) - \varepsilon \). Notice that \( m^*(I) \leq m^*(\bar{I}) \) by monotonicity. So

\[
\ell(I) - \varepsilon < \ell(J) = m^*(J) \text{ by (1), since } J \text{ is a closed bounded interval} \leq m^*(I) \text{ by monotonicity (Lemma 2.2.A)} \leq m^*(\bar{I}) \text{ by monotonicity since } I \subseteq \bar{I} = \ell(\bar{I}) \text{ by (1), since } I \text{ is a closed bounded interval} = \ell(I) \text{ since } I \text{ is a bounded interval}
\]

and therefore \( \ell(I) - \varepsilon < m^*(I) \leq \ell(I) \).
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\[ \sum \ell(I_n) > b_k - a_1. \]

Since \( a_1 < a \) and \( b_k > b \), then \( \sum \ell(I_n) > b - a \). So
\[ m^*(\lbrack a, b \rbrack) = b - a = \ell(\lbrack a, b \rbrack). \]

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\[ \ell(I) - \varepsilon < \ell(J) = m^*(J) \text{ by (1), since } J \text{ is a closed bounded interval} \]

\[ \leq m^*(I) \text{ by monotonicity (Lemma 2.2.A)} \]

\[ \leq m^*(\overline{I}) \text{ by monotonicity since } I \subset \overline{I} \]

\[ = \ell(\overline{I}) \text{ by (1), since } I \text{ is a closed bounded interval} \]

\[ = \ell(I) \text{ since } I \text{ is a bounded interval} \]

and therefore \( \ell(I) - \varepsilon < m^*(I) \leq \ell(I) \). Since \( \varepsilon \) is arbitrary, \( \ell(I) = m^*(I) \).
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and therefore \( \ell(I) - \varepsilon < m^*(I) \leq \ell(I) \). Since \( \varepsilon \) is arbitrary, \( \ell(I) = m^*(I) \).
Proposition 2.1. The outer measure of an interval is its length.

Proof (continued). (3) If \( I \) is an unbounded interval, then given any natural number \( n \in \mathbb{N} \), there is a closed interval \( J \subset I \) with \( \ell(J) = n \). Hence \( m^*(I) \geq m^*(J) = \ell(J) = n \). Since \( m^*(I) \geq n \) and \( n \in \mathbb{N} \) is arbitrary, then \( m^*(I) = \infty = \ell(I) \). \( \square \)
Proposition 2.2

Proposition 2.2. Outer measure is translation invariant; that is, for any set $A$ and number $y$, $m^*(A + y) = m^*(A)$.

Proof. Suppose $m^*(A) = M < \infty$. Then for all $\varepsilon > 0$ there exist $\{I_n\}_{n=1}^{\infty}$ bounded open intervals such that $A \subset \bigcup I_n$ and $\sum \ell(I_n) < M + \varepsilon$ by Theorem 0.3(b).
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Now let $\{J_n\}$ be a collection of bounded open intervals such that $\bigcup J_n \supset A + y$. Assume that $\sum \ell(J_n) < M$. Then $\{J_n - y\}$ is a covering of $A$ and $\sum \ell(J_n - y) = \sum \ell(J_n) < M$, a contradiction. So $\sum \ell(J_n) \geq M$ and hence $m^*(A + y) \geq M$. So $m^*(A + y) = m^*(A) = M$. 

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Now let $\{J_n\}$ be a collection of bounded open intervals such that $\bigcup J_n \supset A + y$. **ASSUME** that $\sum \ell(J_n) < M$. Then $\{J_n - y\}$ is a covering of $A$ and $\sum \ell(J_n - y) = \sum \ell(J_n) < M$, a CONTRADICTION. So $\sum \ell(J_n) \geq M$ and hence $m^*(A + y) \geq M$. So $m^*(A + y) = m^*(A) = M$. 
**Proposition 2.2.** Outer measure is translation invariant; that is, for any set $A$ and number $y$, $m^*(A + y) = m^*(A)$.

**Proof (continued).** Suppose $m^*(A) = \infty$. Then for any $\{I_n\}_{n=1}^{\infty}$ a set of bounded open intervals such that $A \subset \bigcup I_n$, we must have $\sum \ell(I_n) = \infty$. Consider $A + y$. 


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Proposition 2.3

**Proposition 2.3.** Outer measure is countably subadditive. That is, if \( \{E_k\}_{k=1}^{\infty} \) is any countable collection of sets then

\[
m^* \left( \bigcup_{k=1}^{\infty} E_k \right) \leq \sum_{k=1}^{\infty} m^*(E_k).\]

**Proof.** The result holds trivially if \( m^*(E_k) = \infty \) for some \( k \). So without loss of generality assume each \( E_k \) has finite outer measure.
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Proposition 2.3. Outer measure is countably subadditive. That is, if \( \{E_k\}_{k=1}^{\infty} \) is any countable collection of sets then

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\]

Therefore \( m^* \left( \bigcup_{k=1}^{\infty} E_k \right) \leq \sum_{k=1}^{\infty} m^*(E_k). \)
Proposition 2.3. Outer measure is countably subadditive. That is, if \( \{ E_k \}_{k=1}^{\infty} \) is any countable collection of sets then

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Proof. The result holds trivially if \( m^*(E_k) = \infty \) for some \( k \). So without loss of generality assume each \( E_k \) has finite outer measure. Then for all \( \varepsilon > 0 \) and for each \( k \in \mathbb{N} \), there is a countable set of open intervals \( \{ I_{k,i} \}_{i=1}^{\infty} \) such that \( E_k \subset \bigcup_{i=1}^{\infty} I_{k,i} \) and \( \sum_{i=1}^{\infty} \ell(I_{k,i}) < m^*(E_k) + \varepsilon/2^k \) (by Theorem 0.3(b)). Then \( \{ I_{k,i} \} \) where \( i, k \in \mathbb{N} \) is a countable collection (by Theorem 0.10) of open intervals that covers \( \bigcup E_k \). So

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m^*(\bigcup E_k) \leq \sum_{i,k} \ell(I_{k,i}) = \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} \ell(I_{k,i}) < \sum_{k=1}^{\infty} (m^*(E_k) + \varepsilon/2^k) = (\sum_{k=1}^{\infty} m^*(E_k)) + \varepsilon.
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Therefore \( m^*(\bigcup_{k=1}^{\infty} E_k) \leq \sum_{k=1}^{\infty} m^*(E_k) \).