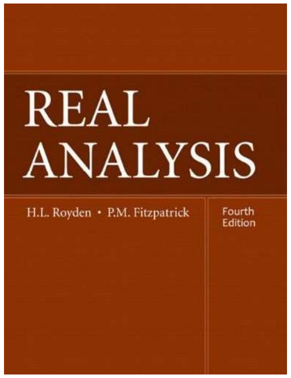
Real Analysis

Chapter 2. Lebesgue Measure

2.3. The σ-Algebra of Lebesgue Measurable Sets—Proofs of Theorems



Proposition 2.4

Proposition 2.4. If $m^*(E) = 0$, then E is measurable.

Proof. Let $A \subset \mathbb{R}$. Then $A \cap E \subset E$ and $A \cap E^c \subset A$. So by monotonicity (Lemma 2.2.A), $m^*(A \cap E) \le m^*(E) = 0$ and $m^*(A \cap E^c) \le m^*(A)$. Therefore

$$m^*(A) \ge m^*(A \cap E^c) = m^*(A \cap E) + m^*(A \cap E^c) = m^*(A \cap E) + m^*(A \setminus E).$$

The reversal of this inequality follows from Note 2.3.A. Hence, E is measurable, as claimed. □

Proposition 2.5

Proposition 2.5. The union of a finite collection of measurable sets is measurable.

Proof. We show the result for two measurable sets, and the general result will follow be induction. Let $E_1, E_2 \in \mathcal{M}$ and let $A \subset \mathbb{R}$. Then

$$\begin{aligned} m^*(A) &= m^*(A \cap E_1) + m^*(A \cap E_1^c), \text{ since } E_1 \in \mathcal{M} \\ &= m^*(A \cap E_1) + \{m^*(\underbrace{[A \cap E_1^c]}_{\subset \mathbb{R}} \cap E_2) + m^*(\underbrace{[A \cap E_1^c]}_{\subset \mathbb{R}} \cap E_2^c)\}, \quad (*) \end{aligned}$$

since $E_2 \in \mathcal{M}$.

Next, in general,

$$[A \cap E_1^c] \cap E_2^c = A \cap [E_1^c \cap E_2^c] = A \cap [E_1 \cup E_2]^c. \quad (**)$$

Proposition 2.5 (continued 1)

We now establish $[A \cap E_1] \cup [A \cap E_1^c \cap E_2] = A \cap (E_1 \cup E_2)$.

(1) Let $x \in [A \cap E_1] \cup [A \cap E_1^c \cap E_2]$.
(a) If $x \in A \cap E_1$ then $x \in A \cap (E_1 \cup E_2)$.
(b) If $x \in A \cap E_1^c \cap E_2$ then $x \in A \cap E_2$ and $x \in A \cap (E_1 \cup E_2)$.
So $[A \cap E_1] \cup [A \cap E_1^c \cap E_2] \subset A \cap (E_1 \cup E_2)$.

(2) Let $x \in A \cap (E_1 \cup E_2) = (A \cap E_1) \cup (A \cap E_2)$.
(a) If $x \in A \cap E_1$ then $x \in [A \cap E_1] \cup [A \cap E_1^c \cap E_2]$.
(b) If $x \in A \cap E_2$ then either $x \in E_1$ or $x \in E_1^c$ and so $x \in A \cap E_2 \cap E_1$ or $x \in A \cap E_2 \cap E_1^c$. Also, $A \cap E_2 \cap E_1 \subset A \cap E_1$ and hence $x \in [A \cap E_2 \cap E_1] \cup [A \cap E_2 \cap E_1^c] \subset [A \cap E_1] \cup [A \cap E_2 \cap E_1^c]$. So $A \cap (E_1 \cup E_2) \subset [A \cap E_1] \cup [A \cap E_2 \cap E_1^c]$.

So $[A \cap E_1] \cup [A \cap E_1^c \cap E_2] = A \cap (E_1 \cup E_2)$.

Proposition 2.5 (continued 2)

Proposition 2.5. The union of a finite collection of measurable sets is measurable.

Proof (continued). Since $[A \cap E_1] \cup [A \cap E_1^c \cap E_2] = A \cap (E_1 \cup E_2)$, by subadditivity (Proposition 2.3),

$$m^*(A \cap [E_1 \cup E_2]) \le m^*(A \cap E_1) + m^*([A \cap E_1^c] \cap E_2). \quad (***)$$

Therefore

$$\begin{aligned} m^*(A) &= m^*(A \cap E_1) + m^*([A \cap E_1^c] \cap E_2) + m^*([A \cap E_1^c] \cap E_2^c) \text{ by } (*) \\ &= m^*(A \cap E_1) + m^*([A \cap E_1^c] \cap E_2) + m^*(A \cap [E_1 \cup E_2]^c) \text{ by } (**) \\ &\ge m^*(A \cap (E_1 \cup E_2)) + m^*(A \cap [E_1 \cup E_2]^c) \text{ by } (***). \end{aligned}$$

The reversal of this inequality follows from Note 2.3.A. Hence $E_1 \cup E_2 \in \mathcal{M}$, as claimed. □

Proposition 2.6

Proposition 2.6. Let $A \subset \mathbb{R}$ and let $\{E_k\}_{k=1}^n$ be a finite disjoint collection of measurable sets. Then $m^*\left(A \cap \left[\bigcup_{k=1}^n E_k\right]\right) = \sum_{k=1}^n m^*(A \cap E_k)$. In particular, when $A = \mathbb{R}$ we see that m^* is finite additive on $\mathcal{M}$.

Proof. We establish the result for $n = 2$ and the general case follows by induction. Let $E_1, E_2 \in \mathcal{M}$, $E_1 \cap E_2 = \varnothing$ and $A \subset \mathbb{R}$. Then $A \cap (E_1 \cup E_2) \cap E_2 = A \cap E_2$ and $A \cap (E_1 \cup E_2) \cap E_2^c = A \cap E_1$. Therefore

$$\begin{aligned} m^*(\underbrace{A \cap (E_1 \cup E_2)}_{\subset \mathbb{R}}) &= m^*([\underbrace{A \cap (E_1 \cup E_2)}_{\subset \mathbb{R}}] \cap E_2) \\ &\quad + m^*([\underbrace{A \cap (E_1 \cup E_2)}_{\subset \mathbb{R}}] \cap E_2^c) \text{ since } E_2 \in \mathcal{M} \\ &= m^*(A \cap E_2) + m^*(A \cap E_1) = \sum_{k=1}^2 m^*(A \cap E_k). \end{aligned}$$

□

Proposition 2.7

Proposition 2.7. The union of a countable collection of measurable sets is measurable.

Proof. Let $\{A_k\}_{k=1}^\infty \subset \mathcal{M}$. Define $E_k = A_k \setminus \cup_{i=1}^{k-1} A_i$. Notice that $\cup_{k=1}^\infty E_k = \cup_{k=1}^\infty A_k$. Then the set $\{E_k\}$ consists of pairwise disjoint sets and since each $A_k \in \mathcal{M}$, then $\cup_{i=1}^{k-1} A_i \in \mathcal{M}$ by Proposition 2.5 and $E_k = A_k \setminus \cup_{i=1}^{k-1} A_k = A_k \cap \left(\cup_{i=1}^{k-1} A_k\right)^c \in \mathcal{M}$ since $\mathcal{M}$ is closed under complements (notice that $\mathcal{M}$ is closed under finite intersections by DeMorgan's Laws; in fact, $\mathcal{M}$ is an algebra).

Let $A \subset \mathbb{R}$ and $n \in \mathbb{N}$. Define $F_n = \cup_{k=1}^n E_k$. Then $F_n \in \mathcal{M}$ by Proposition 2.5, $F_n^c \supset (\cup_{k=1}^\infty E_k)^c$, and so by monotonicity (Lemma 2.2.A)

$$m^*(A) = m^*(A \cap F_n) + m^*(A \cap F_n^c) \ge m^*(A \cap F_n) + m^*(A \cap (\cup_{k=1}^\infty E_k)^c). \quad (*)$$

By Proposition 2.6,

$$m^*(A \cap F_n) = \sum_{k=1}^n m^*(A \cap E_k). \quad (**)$$

Proposition 2.7 (continued)

Proof (continued). So by $(*)$ and $(**)$ we have

$$m^*(A) \ge \left(\sum_{k=1}^n m^*(A \cap E_k)\right) + m^*(A \cap (\cup_{k=1}^\infty E_k)^c)$$

for all $n \in \mathbb{N}$. Therefore

$$\begin{aligned} m^*(A) &\ge \sum_{k=1}^\infty m^*(A \cap E_k) + m^*(A \cap (\cup_{k=1}^\infty E_k)^c) \\ &\ge m^*(\cup_{k=1}^\infty (A \cap E_k)) + m^*(A \cap (\cup_{k=1}^\infty E_k)^c) \\ &\quad \text{by countable subadditivity (Proposition 2.3)} \\ &= m^*(A \cap (\cup_{k=1}^\infty E_k)) + m^*(A \cap (\cup_{k=1}^\infty E_k)^c). \end{aligned}$$

The reversal of this inequality follows from Note 2.3.A. Hence $\cup_{k=1}^\infty E_k = \cup_{k=1}^\infty A_k \in \mathcal{M}$, as claimed. □

Proposition 2.13

Proposition 2.13. (From Section 2.5.) If $\{E_k\}_{k=1}^\infty \subset \mathcal{M}$ and the E_k are pairwise disjoint, then $m^*\left(\dot{\bigcup}_{k=1}^\infty E_k\right) = \sum_{k=1}^\infty m^*(E_k)$.

Proof. First, $\cup_{k=1}^\infty E_k \in \mathcal{M}$ by Proposition 2.7. By countable subadditivity (Proposition 2.3),

$$m^*(\cup_{k=1}^\infty E_k) \le \sum_{k=1}^\infty m^*(E_k). \quad (*)$$

Proposition 2.6 shows that m^* is finite additive on $\mathcal{M}$, and so for all $n \in \mathbb{N}$, $m^*(\cup_{k=1}^n E_k) = \sum_{k=1}^n m^*(E_k)$. By monotonicity (Lemma 2.2.A) $m^*(\cup_{k=1}^\infty E_k) \ge m^*(\cup_{k=1}^n E_k) = \sum_{k=1}^n m^*(E_k)$ for all $n \in \mathbb{N}$, and so

$$m^*(\cup_{k=1}^\infty E_k) \ge \sum_{k=1}^\infty m^*(E_k). \quad (**)$$

Combining $(*)$ and $(**)$ yields the result. □

Proposition 2.8

Proposition 2.8. Every interval is measurable.

Proof. Notice that Exercise 2.11 establishes: If a σ-algebra of subsets of $\mathbb{R}$ contains intervals of the form (a,∞), then it contains all intervals. So we need to only show for all $a \in \mathbb{R}$ that $(a,\infty) \in \mathcal{M}$. Let $A \subset \mathbb{R}$. Without loss of generality $a \notin A$ (otherwise, replace A be $A \setminus \{a\}$ and all relevant outer measures are unchanged; this follows from Exercise 2.9). Define $A_1 = A \cap (-\infty, a] = A \cap (-\infty, a)$ and $A_2 = A \cap (a,\infty)$. Then $A = A_1 \cup A_2$ and $m^*(A) \le m^*(A_1) + m^*(A_2)$ by subadditivity (Proposition 2.3). Let $\{I_k\}$ be a countable open cover of A with bounded open intervals. Define $I_k' = I_k \cap (-\infty, a)$ and $I_k'' = I_k \cap (a,\infty)$. Then $\{I_k'\}$ and $\{I_k''\}$ are countable open covers of A_1 and A_2, respectively, where $\ell(I_k) = \ell(I_k') + \ell(I_k'')$ for all k. Therefore, $m^*(A_1) + m^*(A_2) \le \sum \ell(I_k') + \sum \ell(I_k'') = \sum \ell(I_k)$. Since $\{I_k\}$ was an arbitrary cover of set A, we have $m^*(A_1) + m^*(A_2) \le m^*(A)$. Therefore $m^*(A) = m^*(A_1) + m^*(A_2) = m^*(A \cap (-\infty, a]) + m^*(A \cap (a,\infty))$ and $(a,\infty) \in \mathcal{M}$. □

Proposition 2.10

Proposition 2.10. The translate of a measurable set is measurable.

Proof. Let $E \in \mathcal{M}$, $y \in \mathbb{R}$, and $A \subset \mathbb{R}$. Then

$$\begin{aligned} m^*(A) &= m^*(A - y) \text{ by the translation invariance} \\ &\quad \text{ of } m^* \text{ (Proposition 2.2)} \\ &= m^*([A-y] \cap E) + m^*([A-y] \cap E^c) \text{ because } E \in \mathcal{M} \\ &= m^*(A \cap [E+y]) + m^*(A \cap [E+y]^c), \end{aligned}$$

where the last equality holds because $([A-y] \cap E) + y = A \cap [E+y]$, $([A-y] \cap E^c) + y = A \cap [E+y]^c$, and the fact that m^* is translation invariant (Proposition 2.2). So $E + y \in \mathcal{M}$. □