Chapter 2. Lebesgue Measure
2.3. The $\sigma$-Algebra of Lebesgue Measurable Sets—Proofs of Theorems
Proposition 2.4

Proposition 2.4. If $m^*(E) = 0$, then $E$ is measurable.

Proof. Let $A \subset \mathbb{R}$. Then $A \cap E \subset E$ and $A \cap E^c \subset A$. 
Proposition 2.4. If $m^*(E) = 0$, then $E$ is measurable.

Proof. Let $A \subset \mathbb{R}$. Then $A \cap E \subset E$ and $A \cap E^c \subset A$. So by monotonicity (Lemma 2.2.A), $m^*(A \cap E) \leq m^*(E) = 0$ and $m^*(A \cap E^c) \leq m^*(A)$. (1)
Proposition 2.4

Proposition 2.4. If $m^*(E) = 0$, then $E$ is measurable.

Proof. Let $A \subset \mathbb{R}$. Then $A \cap E \subset E$ and $A \cap E^c \subset A$. So by monotonicity (Lemma 2.2.A), $m^*(A \cap E) \leq m^*(E) = 0$ and $m^*(A \cap E^c) \leq m^*(A)$. Therefore

$$m^*(A) \geq m^*(A \cap E^c) = m^*(A \cap E) + m^*(A \cap E^c) = m^*(A \cap E) + m^*(A \setminus E).$$

The reversal of this inequality (and hence the conclusion of equality) follow from subadditivity, Proposition 2.3. See the comment on page 35 of the text.
Proposition 2.4. If $m^*(E) = 0$, then $E$ is measurable.

**Proof.** Let $A \subset \mathbb{R}$. Then $A \cap E \subset E$ and $A \cap E^c \subset A$. So by monotonicity (Lemma 2.2.A), $m^*(A \cap E) \leq m^*(E) = 0$ and $m^*(A \cap E^c) \leq m^*(A)$. Therefore

$$m^*(A) \geq m^*(A \cap E^c) = m^*(A \cap E) + m^*(A \cap E^c) = m^*(A \cap E) + m^*(A \setminus E).$$

The reversal of this inequality (and hence the conclusion of equality) follow from subadditivity, Proposition 2.3. See the comment on page 35 of the text.
Proposition 2.5

**Proposition 2.5.** The union of a finite collection of measurable sets is measurable.

**Proof.** We show the result for two measurable sets, and the general result will follow be induction. Let \( E_1, E_2 \in \mathcal{M} \) and let \( A \subset \mathbb{R} \).
Proposition 2.5. The union of a finite collection of measurable sets is measurable.

Proof. We show the result for two measurable sets, and the general result will follow by induction. Let $E_1, E_2 \in \mathcal{M}$ and let $A \subset \mathbb{R}$. Then

$$m^*(A) = m^*(A \cap E_1) + m^*(A \cap E_1^c), \text{ since } E_1 \in \mathcal{M}$$

$$= m^*(A \cap E_1) + \{m^*(A \cap E_1^c \cap E_2) + m^*(A \cap E_1^c \cap E_2^c)\}, \quad (*)$$

$$\quad \text{since } E_2 \in \mathcal{M}.$$
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Proposition 2.5. The union of a finite collection of measurable sets is measurable.

Proof. We show the result for two measurable sets, and the general result will follow by induction. Let $E_1, E_2 \in \mathcal{M}$ and let $A \subset \mathbb{R}$. Then

\[
m^*(A) = m^*(A \cap E_1) + m^*(A \cap E_1^c), \text{ since } E_1 \in \mathcal{M}
\]

\[
= m^*(A \cap E_1) + \{ m^*( [A \cap E_1^c] \cap E_2 ) + m^*( [A \cap E_1^c] \cap E_2^c ) \}, \quad (*)
\]

since $E_2 \in \mathcal{M}$.

Next, in general,

\[
[A \cap E_1^c] \cap E_2^c = A \cap [E_1^c \cap E_2^c] = A \cap [E_1 \cup E_2]^c. \quad (**)\]

Proposition 2.5. The union of a finite collection of measurable sets is measurable.

Proof. We show the result for two measurable sets, and the general result will follow by induction. Let $E_1, E_2 \in \mathcal{M}$ and let $A \subset \mathbb{R}$. Then

$$m^*(A) = m^*(A \cap E_1) + m^*(A \cap E_1^c), \quad \text{since } E_1 \in \mathcal{M}$$

$$= m^*(A \cap E_1) + \{m^*([A \cap E_1^c] \cap E_2) + m^*([A \cap E_1^c] \cap E_2^c)\}, \quad (*)$$

since $E_2 \in \mathcal{M}$.

Next, in general,

$$[A \cap E_1^c] \cap E_2^c = A \cap [E_1^c \cap E_2^c] = A \cap [E_1 \cup E_2]^c. \quad (**)$$
Proposition 2.5 (continued 1)

We now establish \([A \cap E_1] \cup [A \cap E_1^c \cap E_2] = A \cap (E_1 \cup E_2)\).

(1) Let \(x \in [A \cap E_1] \cup [A \cap E_1^c \cap E_2]\).
We now establish \([A \cap E_1] \cup [A \cap E_1^c \cap E_2] = A \cap (E_1 \cup E_2)\).

(1) Let \(x \in [A \cap E_1] \cup [A \cap E_1^c \cap E_2]\).

(a) If \(x \in A \cap E_1\) then \(x \in A \cap (E_1 \cup E_2)\).
Proposition 2.5 (continued 1)

We now establish \([A \cap E_1] \cup [A \cap E_1^c \cap E_2] = A \cap (E_1 \cup E_2)\).

(1) Let \(x \in [A \cap E_1] \cup [A \cap E_1^c \cap E_2]\).
   (a) If \(x \in A \cap E_1\) then \(x \in A \cap (E_1 \cup E_2)\).
   (b) If \(x \in A \cap E_1^c \cap E_2\) then \(x \in A \cap E_2\) and \(x \in A \cap (E_1 \cup E_2)\).
Proposition 2.5 (continued 1)

We now establish \([A \cap E_1] \cup [A \cap E_1^c \cap E_2] = A \cap (E_1 \cup E_2)\).

(1) Let \(x \in [A \cap E_1] \cup [A \cap E_1^c \cap E_2]\).
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(b) If \(x \in A \cap E_1^c \cap E_2\) then \(x \in A \cap E_2\) and \(x \in A \cap (E_1 \cup E_2)\).

So \([A \cap E_1] \cup [A \cap E_1^c \cap E_2] \subset A \cap (E_1 \cup E_2)\).
We now establish \([A \cap E_1] \cup [A \cap E_1^c \cap E_2] = A \cap (E_1 \cup E_2)\).

(1) Let \(x \in [A \cap E_1] \cup [A \cap E_1^c \cap E_2]\).
(a) If \(x \in A \cap E_1\) then \(x \in A \cap (E_1 \cup E_2)\).
(b) If \(x \in A \cap E_1^c \cap E_2\) then \(x \in A \cap E_2\) and \(x \in A \cap (E_1 \cup E_2)\).
So \([A \cap E_1] \cup [A \cap E_1^c \cap E_2] \subset A \cap (E_1 \cup E_2)\).

(2) Let \(x \in A \cap (E_1 \cup E_2) = (A \cap E_1) \cup (A \cap E_2)\).
Proposition 2.5 (continued 1)

We now establish \([A \cap E_1] \cup [A \cap E_1^c \cap E_2] = A \cap (E_1 \cup E_2)\).

(1) Let \(x \in [A \cap E_1] \cup [A \cap E_1^c \cap E_2]\).

(a) If \(x \in A \cap E_1\) then \(x \in A \cap (E_1 \cup E_2)\).

(b) If \(x \in A \cap E_1^c \cap E_2\) then \(x \in A \cap E_2\) and \(x \in A \cap (E_1 \cup E_2)\).

So \([A \cap E_1] \cup [A \cap E_1^c \cap E_2] \subset A \cap (E_1 \cup E_2)\).

(2) Let \(x \in A \cap (E_1 \cup E_2) = (A \cap E_1) \cup (A \cap E_2)\).

(a) If \(x \in A \cap E_1\) then \(x \in [A \cap E_1] \cup [A \cap E_1^c \cap E_2]\).
We now establish \[ A \cap E_1 \cup [A \cap E_1^c \cap E_2] = A \cap (E_1 \cup E_2). \]

(1) Let \( x \in [A \cap E_1] \cup [A \cap E_1^c \cap E_2] \).

(a) If \( x \in A \cap E_1 \) then \( x \in A \cap (E_1 \cup E_2) \).

(b) If \( x \in A \cap E_1^c \cap E_2 \) then \( x \in A \cap E_2 \) and \( x \in A \cap (E_1 \cup E_2) \).

So \([A \cap E_1] \cup [A \cap E_1^c \cap E_2] \subset A \cap (E_1 \cup E_2)\).

(2) Let \( x \in A \cap (E_1 \cup E_2) = (A \cap E_1) \cup (A \cap E_2) \).

(a) If \( x \in A \cap E_1 \) then \( x \in [A \cap E_1] \cup [A \cap E_1^c \cap E_2] \).

(b) If \( x \in A \cap E_2 \) then either \( x \in E_1 \) or \( x \in E_1^c \) and so \( x \in A \cap E_2 \cap E_1 \) or \( x \in A \cap E_2 \cap E_1^c \).
We now establish $[A \cap E_1] \cup [A \cap E_1^c \cap E_2] = A \cap (E_1 \cup E_2)$.

(1) Let $x \in [A \cap E_1] \cup [A \cap E_1^c \cap E_2]$.
(a) If $x \in A \cap E_1$ then $x \in A \cap (E_1 \cup E_2)$.
(b) If $x \in A \cap E_1^c \cap E_2$ then $x \in A \cap E_2$ and $x \in A \cap (E_1 \cup E_2)$.
So $[A \cap E_1] \cup [A \cap E_1^c \cap E_2] \subset A \cap (E_1 \cup E_2)$.

(2) Let $x \in A \cap (E_1 \cup E_2) = (A \cap E_1) \cup (A \cap E_2)$.
(a) If $x \in A \cap E_1$ then $x \in [A \cap E_1] \cup [A \cap E_1^c \cap E_2]$.
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Proposition 2.5 (continued 1)

We now establish $[A \cap E_1] \cup [A \cap E_1^c \cap E_2] = A \cap (E_1 \cup E_2)$.

(1) Let $x \in [A \cap E_1] \cup [A \cap E_1^c \cap E_2]$.
   (a) If $x \in A \cap E_1$ then $x \in A \cap (E_1 \cup E_2)$.
   (b) If $x \in A \cap E_1^c \cap E_2$ then $x \in A \cap E_2$ and $x \in A \cap (E_1 \cup E_2)$.

So $[A \cap E_1] \cup [A \cap E_1^c \cap E_2] \subset A \cap (E_1 \cup E_2)$.

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   (a) If $x \in A \cap E_1$ then $x \in [A \cap E_1] \cup [A \cap E_1^c \cap E_2]$.
   (b) If $x \in A \cap E_2$ then either $x \in E_1$ or $x \in E_1^c$ and so $x \in A \cap E_2 \cap E_1$ or $x \in A \cap E_2 \cap E_1^c$. Also, $A \cap E_2 \cap E_1 \subset A \cap E_2$ and hence $x \in [A \cap E_2 \cap E_1] \cup [A \cap E_2 \cap E_1^c] \subset [A \cap E_1] \cup [A \cap E_2 \cap E_1^c]$. So $A \cap (E_1 \cup E_2) \subset [A \cap E_1] \cup [A \cap E_2 \cap E_1^c]$. 

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We now establish \([A \cap E_1] \cup [A \cap E_1^c \cap E_2] = A \cap (E_1 \cup E_2)\).

(1) Let \(x \in [A \cap E_1] \cup [A \cap E_1^c \cap E_2]\).
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So \([A \cap E_1] \cup [A \cap E_1^c \cap E_2] \subset A \cap (E_1 \cup E_2)\).

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(b) If \(x \in A \cap E_2\) then either \(x \in E_1\) or \(x \in E_1^c\) and so \(x \in A \cap E_2 \cap E_1\) or \(x \in A \cap E_2 \cap E_1^c\). Also, \(A \cap E_2 \cap E_1 \subset A \cap E_2\) and hence \(x \in [A \cap E_2 \cap E_1] \cup [A \cap E_2 \cap E_1^c] \subset [A \cap E_1] \cup [A \cap E_2 \cap E_1^c]\). So \(A \cap (E_1 \cup E_2) \subset [A \cap E_1] \cup [A \cap E_2 \cap E_1^c]\).
So \([A \cap E_1] \cup [A \cap E_1^c \cap E_2] = A \cap (E_1 \cup E_2)\).
Proposition 2.5 (continued 1)

We now establish \([A \cap E_1] \cup [A \cap E_1^c \cap E_2] = A \cap (E_1 \cup E_2)\).

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   So \([A \cap E_1] \cup [A \cap E_1^c \cap E_2] \subset A \cap (E_1 \cup E_2)\).

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   (a) If \(x \in A \cap E_1\) then \(x \in [A \cap E_1] \cup [A \cap E_1^c \cap E_2]\).
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   So \([A \cap E_1] \cup [A \cap E_1^c \cap E_2] = A \cap (E_1 \cup E_2)\).
Proposition 2.5 (continued 2)

Proposition 2.5. The union of a finite collection of measurable sets is measurable.

Proof (continued). Since \([A \cap E_1] \cup [A \cap E_1^c \cap E_2] = A \cap (E_1 \cup E_2)\), by subadditivity (Proposition 2.3),

\[
m^*(A \cap [E_1 \cup E_2]) \leq m^*(A \cap E_1) + m^*([A \cap E_1^c] \cap E_2).
\] (* * *)

Therefore

\[
m^*(A) = m^*(A \cap E_1) + m^*([A \cap E_1^c] \cap E_2) + m^*(A \cap (E_1 \cup E_2)) \leq m^*(A \cap E_1) + m^*([A \cap E_1^c] \cap E_2) + m^*(A \cap (E_1 \cup E_2))
\]

The reversal of this inequality (and hence the conclusion of equality) follows from subadditivity, Proposition 2.3. See the comment on page 35 of the text. Hence \(E_1 \cup E_2 \in M\).
Proposition 2.5. The union of a finite collection of measurable sets is measurable.

Proof (continued). Since \([A \cap E_1] \cup [A \cap E_1^c \cap E_2] = A \cap (E_1 \cup E_2)\), by subadditivity (Proposition 2.3),

\[
m^*(A \cap [E_1 \cup E_2]) \leq m^*(A \cap E_1) + m^*([A \cap E_1^c] \cap E_2).
\] \((***)\)

Therefore

\[
m^*(A) = m^*(A \cap E_1) + m^*([A \cap E_1^c] \cap E_2) + m^*([A \cap E_1^c] \cap E_2^c) \text{ by (**) and (***)}
\]

\[
m^*(A) = m^*(A \cap E_1) + m^*([A \cap E_1^c] \cap E_2) + m^*(A \cap [E_1 \cup E_2]^c) \text{ by (**) and (***)}
\]

\[
m^*(A) \geq m^*(A \cap (E_1 \cup E_2)) + m^*(A \cap [E_1 \cup E_2]^c) \text{ by (***)}.
\]
Proposition 2.5 (continued 2)

Proposition 2.5. The union of a finite collection of measurable sets is measurable.

Proof (continued). Since \([A \cap E_1] \cup [A \cap E_1^c \cap E_2] = A \cap (E_1 \cup E_2)\), by subadditivity (Proposition 2.3),

\[
m^*(A \cap [E_1 \cup E_2]) \leq m^*(A \cap E_1) + m^*([A \cap E_1^c] \cap E_2). \quad (\ast \ast \ast)
\]

Therefore

\[
m^*(A) = m^*(A \cap E_1) + m^*([A \cap E_1^c] \cap E_2) + m^*([A \cap E_1^c] \cap E_2^c) \quad \text{by (\ast)}
\]

\[
= m^*(A \cap E_1) + m^*([A \cap E_1^c] \cap E_2) + m^*(A \cap [E_1 \cup E_2]^c) \quad \text{by (\ast \ast)}
\]

\[
\geq m^*(A \cap (E_1 \cup E_2)) + m^*(A \cap [E_1 \cup E_2]^c) \quad \text{by (\ast \ast \ast)}.
\]

The reversal of this inequality (and hence the conclusion of equality) follows from subadditivity, Proposition 2.3. See the comment on page 35 of the text. Hence \(E_1 \cup E_2 \in \mathcal{M}\).
**Proposition 2.5.** The union of a finite collection of measurable sets is measurable.

**Proof (continued).** Since \([A \cap E_1] \cup [A \cap E_1^c \cap E_2] = A \cap (E_1 \cup E_2)\), by subadditivity (Proposition 2.3),

\[
m^*(A \cap [E_1 \cup E_2]) \leq m^*(A \cap E_1) + m^*([A \cap E_1^c] \cap E_2). \quad (\ast \ast \ast)
\]

Therefore

\[
m^*(A) = m^*(A \cap E_1) + m^*([A \cap E_1^c] \cap E_2) + m^*([A \cap E_1^c] \cap E_2^c) \quad \text{by (\ast)}
\]
\[
= m^*(A \cap E_1) + m^*([A \cap E_1^c] \cap E_2) + m^*(A \cap [E_1 \cup E_2]^c) \quad \text{by (\ast\ast)}
\]
\[
\geq m^*(A \cap (E_1 \cup E_2)) + m^*(A \cap [E_1 \cup E_2]^c) \quad \text{by (\ast \ast \ast)}.
\]

The reversal of this inequality (and hence the conclusion of equality) follows from subadditivity, Proposition 2.3. See the comment on page 35 of the text. Hence \(E_1 \cup E_2 \in \mathcal{M}\). \(\square\)
Proposition 2.6. Let $A \subset \mathbb{R}$ and let $\{E_k\}_{k=1}^n$ be a finite disjoint collection of measurable sets. Then $m^* \left( A \cap \left( \bigcup_{k=1}^n E_k \right) \right) = \sum_{k=1}^n m^*(A \cap E_k)$. In particular, when $A = \mathbb{R}$ we see that $m^*$ is finite additive on $\mathcal{M}$.

Proof. We establish the result for $n = 2$ and the general case follows by induction. Let $E_1, E_2 \in \mathcal{M}$, $E_1 \cap E_2 = \emptyset$ and $A \subset \mathbb{R}$. 
Proposition 2.6

**Proposition 2.6.** Let $A \subset \mathbb{R}$ and let $\{E_k\}_{k=1}^n$ be a finite disjoint collection of measurable sets. Then $m^* \left( A \cap \left( \bigcup_{k=1}^n E_k \right) \right) = \sum_{k=1}^n m^*(A \cap E_k)$. In particular, when $A = \mathbb{R}$ we see that $m^*$ is finite additive on $\mathcal{M}$.

**Proof.** We establish the result for $n = 2$ and the general case follows by induction. Let $E_1, E_2 \in \mathcal{M}$, $E_1 \cap E_2 = \emptyset$ and $A \subset \mathbb{R}$. Then $A \cap (E_1 \cup E_2) \cap E_2 = A \cap E_2$ and $A \cap (E_1 \cup E_2) \cap E_2^c = A \cap E_1$. Therefore

$$m^*(A \cap (E_1 \cup E_2)) \subset \mathbb{R} = m^*([A \cap (E_1 \cup E_2)] \cap E_2) \subset \mathbb{R} + m^*([A \cap (E_1 \cup E_2)] \cap E_2^c) \subset \mathbb{R}$$

since $E_2 \in \mathcal{M}$

$$= m^*(A \cap E_2) + m^*(A \cap E_1) = \sum_{k=1}^2 m^*(A \cap E_k). \qed$$
Proposition 2.6. Let \( A \subset \mathbb{R} \) and let \( \{E_k\}_{k=1}^n \) be a finite disjoint collection of measurable sets. Then 
\[
 m^*(A \cap \left[ \bigcup_{k=1}^n E_k \right]) = \sum_{k=1}^n m^*(A \cap E_k).
\]
In particular, when \( A = \mathbb{R} \) we see that \( m^* \) is finite additive on \( \mathcal{M} \).

Proof. We establish the result for \( n = 2 \) and the general case follows by induction. Let \( E_1, E_2 \in \mathcal{M} \), \( E_1 \cap E_2 = \emptyset \) and \( A \subset \mathbb{R} \). Then 
\[
 A \cap (E_1 \cup E_2) \cap E_2 = A \cap E_2 \quad \text{and} \quad A \cap (E_1 \cup E_2) \cap E_2^c = A \cap E_1.
\]
Therefore
\[
 m^*(A \cap (E_1 \cup E_2)) = m^*(A \cap (E_1 \cup E_2) \cap E_2) + m^*(A \cap (E_1 \cup E_2) \cap E_2^c) \quad \text{since} \quad E_2 \in \mathcal{M}
\]
\[
 = m^*(A \cap E_2) + m^*(A \cap E_1) = \sum_{k=1}^2 m^*(A \cap E_k). \quad \Box
\]
Proposition 2.7

Proposition 2.7. The union of a countable collection of measurable sets is measurable.

Proof. Let \( \{A_k\}_{k=1}^{\infty} \subset \mathcal{M} \).
Proposition 2.7

Proposition 2.7. The union of a countable collection of measurable sets is measurable.

Proof. Let \( \{A_k\}_{k=1}^{\infty} \subset \mathcal{M} \). Define \( E_k = A_k \setminus \bigcup_{i=1}^{k-1} A_i \). Notice that \( \bigcup_{k=1}^{\infty} E_k = \bigcup_{k=1}^{\infty} A_k \). Then the set \( \{E_k\} \) consists of pairwise disjoint sets and since each \( A_k \in \mathcal{M} \), then \( \bigcup_{i=1}^{k-1} A_i \in \mathcal{M} \) by Proposition 2.5 and \( E_k = A_k \setminus \bigcup_{i=1}^{k-1} A_k = A_k \cap \left( \bigcup_{i=1}^{k-1} A_k \right)^c \in \mathcal{M} \) since \( \mathcal{M} \) is closed under complements (notice that \( \mathcal{M} \) is closed under finite intersections by DeMorgan’s Laws; in fact, \( \mathcal{M} \) is an algebra).
Proposition 2.7. The union of a countable collection of measurable sets is measurable.

Proof. Let \( \{A_k\}_{k=1}^{\infty} \subset \mathcal{M} \). Define \( E_k = A_k \setminus \bigcup_{i=1}^{k-1} A_i \). Notice that \( \bigcup_{k=1}^{\infty} E_k = \bigcup_{k=1}^{\infty} A_k \). Then the set \( \{E_k\} \) consists of pairwise disjoint sets and since each \( A_k \in \mathcal{M} \), then \( \bigcup_{i=1}^{k-1} A_i \in \mathcal{M} \) by Proposition 2.5 and \( E_k = A_k \setminus \bigcup_{i=1}^{k-1} A_k = A_k \cap \left( \bigcup_{i=1}^{k-1} A_k \right)^c \in \mathcal{M} \) since \( \mathcal{M} \) is closed under complements (notice that \( \mathcal{M} \) is closed under finite intersections by DeMorgan’s Laws; in fact, \( \mathcal{M} \) is an algebra).

Let \( A \subset \mathbb{R} \) and \( n \in \mathbb{N} \). Define \( F_n = \bigcup_{k=1}^{n} E_k \).
Proposition 2.7. The union of a countable collection of measurable sets is measurable.

**Proof.** Let \( \{A_k\}_{k=1}^\infty \subset \mathcal{M} \). Define \( E_k = A_k \setminus \bigcup_{i=1}^{k-1} A_i \). Notice that \( \bigcup_{k=1}^\infty E_k = \bigcup_{k=1}^\infty A_k \). Then the set \( \{E_k\} \) consists of pairwise disjoint sets and since each \( A_k \in \mathcal{M} \), then \( \bigcup_{i=1}^{k-1} A_i \in \mathcal{M} \) by Proposition 2.5 and

\[
E_k = A_k \setminus \bigcup_{i=1}^{k-1} A_k = A_k \cap \left( \bigcup_{i=1}^{k-1} A_k \right)^c \in \mathcal{M}
\]

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Let \( A \subset \mathbb{R} \) and \( n \in \mathbb{N} \). Define \( F_n = \bigcup_{k=1}^n E_k \). Then \( F_n \in \mathcal{M} \) by Proposition 2.5, \( F_n^c \supset (\bigcup_{k=1}^\infty E_k)^c \), and so by monotonicity (Lemma 2.2.A)

\[
m^*(A) = m^*(A \cap F_n) + m^*(A \cap F_n^c) \geq m^*(A \cap F_n) + m^*(A \cap (\bigcup_{k=1}^\infty E_k)^c).
\]

(*)
Proposition 2.7

Proposition 2.7. The union of a countable collection of measurable sets is measurable.

Proof. Let \( \{A_k\}_{k=1}^{\infty} \subset \mathcal{M} \). Define \( E_k = A_k \setminus \bigcup_{i=1}^{k-1} A_i \). Notice that \( \bigcup_{k=1}^{\infty} E_k = \bigcup_{k=1}^{\infty} A_k \). Then the set \( \{E_k\} \) consists of pairwise disjoint sets and since each \( A_k \in \mathcal{M} \), then \( \bigcup_{i=1}^{k-1} A_i \in \mathcal{M} \) by Proposition 2.5 and \( E_k = A_k \setminus \bigcup_{i=1}^{k-1} A_k = A_k \cap \left( \bigcup_{i=1}^{k-1} A_k \right)^c \in \mathcal{M} \) since \( \mathcal{M} \) is closed under complements (notice that \( \mathcal{M} \) is closed under finite intersections by DeMorgan’s Laws; in fact, \( \mathcal{M} \) is an algebra).

Let \( A \subset \mathbb{R} \) and \( n \in \mathbb{N} \). Define \( F_n = \bigcup_{k=1}^{n} E_k \). Then \( F_n \in \mathcal{M} \) by Proposition 2.5, \( F_n^c \supset (\bigcup_{k=1}^{\infty} E_k)^c \), and so by monotonicity (Lemma 2.2.A) \( m^*(A) = m^*(A \cap F_n) + m^*(A \cap F_n^c) \geq m^*(A \cap F_n) + m^*(A \cap (\bigcup_{k=1}^{\infty} E_k)^c) \). (*)

By Proposition 2.6,

\[
m^*(A \cap F_n) = \sum_{k=1}^{n} m^*(A \cap E_k). \tag{**}
\]
Proposition 2.7. The union of a countable collection of measurable sets is measurable.

Proof. Let \( \{ A_k \}_{k=1}^\infty \subset \mathcal{M} \). Define \( E_k = A_k \setminus \bigcup_{i=1}^{k-1} A_i \). Notice that \( \bigcup_{k=1}^\infty E_k = \bigcup_{k=1}^\infty A_k \). Then the set \( \{ E_k \} \) consists of pairwise disjoint sets and since each \( A_k \in \mathcal{M} \), then \( \bigcup_{i=1}^{k-1} A_i \in \mathcal{M} \) by Proposition 2.5 and \( E_k = A_k \setminus \bigcup_{i=1}^{k-1} A_k = A_k \cap \left( \bigcup_{i=1}^{k-1} A_k \right)^c \in \mathcal{M} \) since \( \mathcal{M} \) is closed under complements (notice that \( \mathcal{M} \) is closed under finite intersections by DeMorgan’s Laws; in fact, \( \mathcal{M} \) is an algebra).

Let \( A \subset \mathbb{R} \) and \( n \in \mathbb{N} \). Define \( F_n = \bigcup_{k=1}^n E_k \). Then \( F_n \in \mathcal{M} \) by Proposition 2.5, \( F_n^c \supset (\bigcup_{k=1}^\infty E_k)^c \), and so by monotonicity (Lemma 2.2.A) \( m^*(A) = m^*(A \cap F_n) + m^*(A \cap F_n^c) \geq m^*(A \cap F_n) + m^*(A \cap (\bigcup_{k=1}^\infty E_k)^c) \). \((*)\)

By Proposition 2.6,

\[
m^*(A \cap F_n) = \sum_{k=1}^n m^*(A \cap E_k). \quad (**)
\]
Proposition 2.7 (continued)

**Proof (continued).** So by (*) and (***) we have

$$m^*(A) \geq \left( \sum_{k=1}^{n} m^*(A \cap E_k) \right) + m^*(A \cap (\bigcup_{k=1}^{\infty} E_k)^c)$$

for all $n \in \mathbb{N}$. Therefore

$$m^*(A) \geq \sum_{k=1}^{\infty} m^*(A \cap E_k) + m^*(A \cap (\bigcup_{k=1}^{\infty} E_k)^c)$$

by countable subadditivity

$$= m^*(A \cap (\bigcup_{k=1}^{\infty} E_k)) + m^*(A \cap (\bigcup_{k=1}^{\infty} E_k)^c).$$
Proposition 2.7 (continued)

Proof (continued). So by (\ast) and (\ast \ast) we have

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for all \( n \in \mathbb{N} \). Therefore

\[ m^*(A) \geq \sum_{k=1}^{\infty} m^*(A \cap E_k) + m^*(A \cap (\bigcup_{k=1}^{\infty} E_k)^c) \]

\[ \geq m^*(\bigcup_{k=1}^{\infty} (A \cap E_k)) + m^*(A \cap (\bigcup_{k=1}^{\infty} E_k)^c) \]

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\[ = m^*(A \cap (\bigcup_{k=1}^{\infty} E_k)) + m^*(A \cap (\bigcup_{k=1}^{\infty} E_k)^c). \]

The reversal of this inequality (and hence the conclusion of equality) follow from subadditivity, Proposition 2.3. See the comment on page 35 of the text. Hence \( \bigcup_{k=1}^{\infty} E_k = \bigcup_{k=1}^{\infty} A_k \in \mathcal{M} \).
Proposition 2.7 (continued)

Proof (continued). So by (*) and (**) we have

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Proposition 2.13

Proposition 2.13. (From Section 2.5.) If $\{E_k\}_{k=1}^{\infty} \subset \mathcal{M}$ and the $E_k$ are pairwise disjoint, then $m^* \left( \bigcup_{k=1}^{\infty} E_k \right) = \sum_{k=1}^{\infty} m^*(E_k)$.

Proof. First, $\bigcup_{k=1}^{\infty} E_k \in \mathcal{M}$ by Proposition 2.7.
Proposition 2.13

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Proof. First, $\bigcup_{k=1}^{\infty} E_k \in \mathcal{M}$ by Proposition 2.7. By countable subadditivity (Proposition 2.3),

$$m^*(\bigcup_{k=1}^{\infty} E_k) \leq \sum_{k=1}^{\infty} m^*(E_k). \quad (*)$$

Proposition 2.6 shows that $m^*$ is finite additive on $\mathcal{M}$, and so for all $n \in \mathbb{N}$, $m^*(\bigcup_{k=1}^{n} E_k) = \sum_{k=1}^{n} m^*(E_k)$. 


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\]

Proposition 2.6 shows that \( m^* \) is finite additive on \( \mathcal{M} \), and so for all \( n \in \mathbb{N} \), \( m^*(\bigcup_{k=1}^{n} E_k) = \sum_{k=1}^{n} m^*(E_k) \). By monotonicity (Lemma 2.2.A) \( m^*(\bigcup_{k=1}^{\infty} E_k) \geq m^*(\bigcup_{k=1}^{n} E_k) = \sum_{k=1}^{n} m^*(E_k) \) for all \( n \in \mathbb{N} \), and so

\[
m^*(\bigcup_{k=1}^{\infty} E_k) \geq \sum_{k=1}^{\infty} m^*(E_k). \quad (**)\]
Proposition 2.13

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m^* \left( \bigcup_{k=1}^{\infty} E_k \right) = \sum_{k=1}^{\infty} m^*(E_k). \]

Proof. First, \( \bigcup_{k=1}^{\infty} E_k \in \mathcal{M} \) by Proposition 2.7. By countable subadditivity (Proposition 2.3),
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m^*(\bigcup_{k=1}^{\infty} E_k) \leq \sum_{k=1}^{\infty} m^*(E_k). \quad (*)
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Proposition 2.6 shows that \( m^* \) is finite additive on \( \mathcal{M} \), and so for all \( n \in \mathbb{N} \),
\[
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\]
By monotonicity (Lemma 2.2.A) \( m^*(\bigcup_{k=1}^{\infty} E_k) \geq m^*(\bigcup_{k=1}^{n} E_k) = \sum_{k=1}^{n} m^*(E_k) \) for all \( n \in \mathbb{N} \), and so
\[
m^*(\bigcup_{k=1}^{\infty} E_k) \geq \sum_{k=1}^{\infty} m^*(E_k). \quad (***)
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Combining \((*)\) and \((***)\) yields the result.
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Proposition 2.6 shows that \( m^* \) is finite additive on \( \mathcal{M} \), and so for all \( n \in \mathbb{N} \), \( m^*(\bigcup_{k=1}^{n} E_k) = \sum_{k=1}^{n} m^*(E_k) \). By monotonicity (Lemma 2.2.A) \( m^*(\bigcup_{k=1}^{\infty} E_k) \geq m^*(\bigcup_{k=1}^{n} E_k) = \sum_{k=1}^{n} m^*(E_k) \) for all \( n \in \mathbb{N} \), and so

\[
m^*(\bigcup_{k=1}^{\infty} E_k) \geq \sum_{k=1}^{\infty} m^*(E_k). \quad (**)
\]

Combining (*) and (**) yields the result.
Proposition 2.8. Every interval is measurable.

Proof. Notice that Exercise 2.11 establishes: If a σ-algebra of subsets of \( \mathbb{R} \) contains intervals of the form \((a, \infty)\), then it contains all intervals. So we need to only show for all \( a \in \mathbb{R} \) that \((a, \infty) \in \mathcal{M}\).
Proposition 2.8

**Proposition 2.8.** Every interval is measurable.

**Proof.** Notice that Exercise 2.11 establishes: If a $\sigma$-algebra of subsets of $\mathbb{R}$ contains intervals of the form $(a, \infty)$, then it contains all intervals. So we need to only show for all $a \in \mathbb{R}$ that $(a, \infty) \in \mathcal{M}$. Let $A \subset \mathbb{R}$. Without loss of generality $a \not\in A$ (otherwise, replace $A$ by $A \setminus \{a\}$ and all relevant outer measures are unchanged; this follows from Exercise 2.9). Define $A_1 = A \cap (-\infty, a] = A \cap (-\infty, a)$ and $A_2 = A \cap (a, \infty)$. Then $A = A_1 \cup A_2$ and $m^*(A) \leq m^*(A_1) + m^*(A_2)$ by subbadditivity (Proposition 2.3).
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Proposition 2.8

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(*Note: The proof is based on the properties of $\sigma$-algebras and the construction of open covers to show the measurability of intervals within these properties.*
Proposition 2.8

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Proof. Notice that Exercise 2.11 establishes: If a σ-algebra of subsets of \( \mathbb{R} \) contains intervals of the form \((a, \infty)\), then it contains all intervals. So we need to only show for all \( a \in \mathbb{R} \) that \((a, \infty) \in \mathcal{M}\). Let \( A \subset \mathbb{R} \). Without loss of generality \( a \not\in A \) (otherwise, replace \( A \) be \( A \setminus \{a\} \) and all relevant outer measures are unchanged; this follows from Exercise 2.9). Define \( A_1 = A \cap (-\infty, a] = A \cap (-\infty, a) \) and \( A_2 = A \cap (a, \infty) \). Then \( A = A_1 \cup A_2 \) and \( m^*(A) \leq m^*(A_1) + m^*(A_2) \) by subadditivity (Proposition 2.3). Let \( \{I_k\} \) be a countable open cover of \( A \) with (bounded) open intervals. Define \( I'_k = I_k \cap (-\infty, a) \) and \( I''_k = I_k \cap (a, \infty) \). Then \( \{I'_k\} \) and \( \{I''_k\} \) are countable open covers of \( A_1 \) and \( A_2 \), respectively, where \( \ell(I_k) = \ell(I'_k) + \ell(I''_k) \) for all \( k \). Therefore,
\[
m^*(A_1) + m^*(A_2) \leq \sum \ell(I'_k) + \sum \ell(I''_k) = \sum \ell(I_k).\]
Since \( \{I_k\} \) was an arbitrary cover of set \( A \), we have \( m^*(A_1) + m^*(A_2) \leq m^*(A) \). Therefore \( m^*(A) = m^*(A_1) + m^*(A_2) = m^*(A \cap (-\infty, a]) + m^*(A \cap (a, \infty)) \) and \((a, \infty) \in \mathcal{M}\). \( \square \)
Proposition 2.8

Proposition 2.8. Every interval is measurable.

Proof. Notice that Exercise 2.11 establishes: If a $\sigma$-algebra of subsets of $\mathbb{R}$ contains intervals of the form $(a, \infty)$, then it contains all intervals. So we need to only show for all $a \in \mathbb{R}$ that $(a, \infty) \in \mathcal{M}$. Let $A \subset \mathbb{R}$. Without loss of generality $a \notin A$ (otherwise, replace $A$ by $A \setminus \{a\}$ and all relevant outer measures are unchanged; this follows from Exercise 2.9). Define $A_1 = A \cap (-\infty, a] = A \cap (-\infty, a)$ and $A_2 = A \cap (a, \infty)$. Then $A = A_1 \cup A_2$ and $m^*(A) \leq m^*(A_1) + m^*(A_2)$ by subadditivity (Proposition 2.3). Let $\{I_k\}$ be a countable open cover of $A$ with (bounded) open intervals. Define $I'_k = I_k \cap (-\infty, a)$ and $I''_k = I_k \cap (a, \infty)$. Then $\{I'_k\}$ and $\{I''_k\}$ are countable open covers of $A_1$ and $A_2$, respectively, where $\ell(I_k) = \ell(I'_k) + \ell(I''_k)$ for all $k$. Therefore, $m^*(A_1) + m^*(A_2) \leq \sum \ell(I'_k) + \sum \ell(I''_k) = \sum \ell(I_k)$. Since $\{I_k\}$ was an arbitrary cover of set $A$, we have $m^*(A_1) + m^*(A_2) \leq m^*(A)$. Therefore $m^*(A) = m^*(A_1) + m^*(A_2) = m^*(A \cap (-\infty, a]) + m^*(A \cap (a, \infty))$ and $(a, \infty) \in \mathcal{M}$. \qed
Proposition 2.10. The translate of a measurable set is measurable.

Proof. Let $E \in \mathcal{M}$, $y \in \mathbb{R}$, and $A \subset \mathbb{R}$.
Proposition 2.10. The translate of a measurable set is measurable.

Proof. Let $E \in \mathcal{M}$, $y \in \mathbb{R}$, and $A \subset \mathbb{R}$. Then

$$m^*(A) = m^*(A - y)$$

by the translation invariance of $m^*$ (Proposition 2.2)

$$= m^*([A - y] \cap E) + m^*([A - y] \cap E^c)$$

because $E \in \mathcal{M}$

$$= m^*(A \cap [E + y]) + m^*(A \cap [E + y]^c).$$

So $E + y \in \mathcal{M}$. \qed
Proposition 2.10. The translate of a measurable set is measurable.

Proof. Let $E \in \mathcal{M}$, $y \in \mathbb{R}$, and $A \subset \mathbb{R}$. Then

$$m^*(A) = m^*(A - y) \text{ by the translation invariance of } m^* \text{ (Proposition 2.2)}$$

$$= m^*([A - y] \cap E) + m^*([A - y] \cap E^c) \text{ because } E \in \mathcal{M}$$

$$= m^*(A \cap [E + y]) + m^*(A \cap [E + y]^c).$$

So $E + y \in \mathcal{M}.$