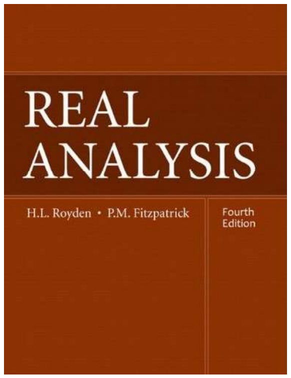
Real Analysis

Chapter 2. Lebesgue Measure

2.4. Outer and Inner Approximation of Lebesgue Measurable Sets—Proofs of Theorems



Lemma 2.2.A. Excision Property

Lemma 2.2.A. The Excision Property.
If A is measurable and $m^*(A) < \infty$ and $A \subset B$ then

$$m^*(B \setminus A) = m^*(B) - m^*(A).$$

Proof. Since A is measurable and $B \subset \mathbb{R}$ then by the definition of measurable,

$$\begin{aligned} m^*(B) &= m^*(B \cap A) + m^*(B \cap A^c) \\ &= m^*(B \cap A) + m^*(B \setminus A) \text{ since } B \cap A^c = B \setminus A \\ &= m^*(A) + m^*(B \setminus A) \text{ since } A \subset B. \end{aligned}$$

Since $m^*(A) < \infty$, then $m^*(B) - m^*(A) = m^*(B \setminus A)$. □

Theorem 2.11

Theorem 2.11. Let $E \subset \mathbb{R}$. Then each of the following are equivalent to the measurability of E:

(i) For each $\varepsilon > 0$, there is an open set $\mathcal{O}$ containing E for which $m^*(\mathcal{O} \setminus E) < \varepsilon$.

(ii) There is a G_δ set G containing E for which $m^*(G \setminus E) = 0$.

(iii) For each $\varepsilon > 0$, there is a closed set F contained in E for which $m^*(E \setminus F) < \varepsilon$.

(iv) There is an F_σ set F contained in E for which $m^*(E \setminus F) = 0$.

Proof. measurable $\Rightarrow$ (i) Let $E \in \mathcal{M}$ and $\varepsilon > 0$. First, suppose $m^*(E) < \infty$. Then from the definition of outer measure, there is an open cover of intervals $\{I_k\}_{k=1}^\infty$ of E for which $\sum \ell(I_k) < m^*(E) + \varepsilon$ (by Theorem 0.3(b)). Define $\mathcal{O} = \cup I_k$. Then $\mathcal{O}$ is open and $E \subset \mathcal{O}$. Also, $m^*(\mathcal{O}) \le \sum \ell(I_k) < m^*(E) + \varepsilon$, or $m^*(\mathcal{O}) - m^*(E) < \varepsilon$. Now

$$m^*(\mathcal{O}) = m^*(\mathcal{O} \cap E) + m^*(\mathcal{O} \cap E^c) \text{ since } E \in \mathcal{M}$$

Theorem 2.11 (continued 1)

Proof (continued).

$$\ldots m^*(\mathcal{O}) = m^*(E) + m^*(\mathcal{O} \setminus E)$$

and since $m^*(E) < \infty$, we have $m^*(\mathcal{O} \setminus E) = m^*(\mathcal{O}) - m^*(E)$. That is, $m^*(\mathcal{O} \setminus E) = m^*(\mathcal{O}) - m^*(E) < \varepsilon$ for all $\varepsilon > 0$. So $E \in \mathcal{M}$ and $m^*(E) < \infty$ implies (i).

Now suppose $m^*(E) = \infty$. Then $E = \cup_{k=1}^\infty E_k$ where each E_k is measurable and of finite measure (say, $E_{2k} = E \cap [k-1, k)$ and $E_{2k+1} = E \cap [-k-1, -k)$). From the above argument, there is open $\mathcal{O}_k \supset E_k$ for which $m^*(\mathcal{O}_k \setminus E_k) < \varepsilon/2^k$. Define $\mathcal{O} = \cup \mathcal{O}_k$. Then $\mathcal{O}$ is open, $\mathcal{O} \supset E$ and $\mathcal{O} \setminus E = \cup \mathcal{O}_k \setminus E \subset \cup(\mathcal{O}_k \setminus E_k)$ and so $m^*(\mathcal{O} \setminus E) \le \sum m^*(\mathcal{O}_k \setminus E_k) < \sum \varepsilon/2^k = \varepsilon$. So $E \in \mathcal{M}$ and $m^*(E) = \infty$ implies (i).

Therefore $E \in \mathcal{M}$ implies (i).

Theorem 2.11 (continued 2)

Proof (continued). (i) $\Rightarrow$ (ii) Suppose (i) holds for E. Then for each $k \in \mathbb{N}$ there is open $\mathcal{O}_k \supset E$ where $m^*(\mathcal{O}_k \setminus E) < 1/k$. Define $G = \cap \mathcal{O}_k$. Then G is G_δ and $G \supset E$. Also, since $G \setminus E \subset \mathcal{O}_k \setminus E$, monotonicity implies $m^*(G \setminus E) \leq m^*(\mathcal{O}_k \setminus E) < 1/k$ for all $k \in \mathbb{N}$. Therefore $m^*(G \setminus E) = 0$ and so (ii) holds.

(ii) $\Rightarrow$ measurable Suppose (ii) holds for E. Then $m^*(G \setminus E) = 0$ and so by Proposition 2.4 $G \setminus E \in \mathcal{M}$ and hence $(G \setminus E)^c \in \mathcal{M}$. Since G is G_δ, then $G \in \mathcal{M}$ and hence $E = G \cap (G \setminus E)^c \in \mathcal{M}$ since $\mathcal{M}$ is a σ-algebra. (So $E \in \mathcal{M}$ implies (i) implies (ii) implies $E \in \mathcal{M}$, and so these three properties are equivalent.)

To show that $E \in \mathcal{M}$ is equivalent to (iii) and (iv) (and hence to (i) and (ii)), we need to apply DeMorgan's Laws and the facts that if $E \in \mathcal{M}$ then $E^c \in \mathcal{M}$, a set is open if and only if its complement is closed, and a set is F_σ if and only if its complement is G_δ (this is Exercise 2.16). □

Theorem 2.12

Theorem 2.12. Let $E \in \mathcal{M}$, $m^*(E) < \infty$. Then for each $\varepsilon > 0$, there is a finite disjoint collection of open intervals $\{I_k\}_{k=1}^n$ for which, if $\mathcal{O} = \cup_{k=1}^n I_k$, then

$$m^*(E \Delta \mathcal{O}) = m^*(E \setminus \mathcal{O}) + m^*(\mathcal{O} \setminus E) < \varepsilon.$$

Proof. Let $\varepsilon > 0$ be given. Since E is measurable, by Theorem 2.11 (the "measurable implies (i)" part), there is an open set $\mathcal{U}$ such that $E \subset \mathcal{U}$ and $m^*(\mathcal{U} \setminus E) < \varepsilon/2$. Since $m^*(E) < \infty$ by hypothesis and $E \subset \mathcal{U}$, by the Excision Property (Lemma 2.4.A) we have $m^*(\mathcal{U} \setminus E) = m^*(\mathcal{U}) - m^*(E)$ and so

$$m^*(\mathcal{U}) = m^*(E) + m^*(\mathcal{U} \setminus E) < m^*(E) + \varepsilon/2.$$

So $m^*(\mathcal{U}) < \infty$. Since $\mathcal{U}$ is an open set of real numbers, then by Theorem 0.7 we have $\mathcal{U} = \cup_{k=1}^\infty I_k$ for some set of open intervals $\{I_k\}_{k=1}^\infty$. Each interval is measurable by Proposition 2.8 and the outer measure of an interval is its length by Proposition 2.1.

Theorem 2.12 (continued)

Proof (continued). Therefore by finite additivity (Proposition 2.6) and the monotonicity of outer measure (Lemma 2.2.A) we have for each $n \in \mathbb{N}$:

$$\sum_{k=1}^n \ell(I_k) = m^*\left(\cup_{k=1}^n I_k\right) \leq m^*(\mathcal{U}) < \infty.$$

Therefore $\sum_{k=1}^\infty \ell(I_k) < \infty$ and so $\{\ell(I_k)\}_{k=1}^\infty$ is a summable sequence of nonnegative real numbers. So by a property of summable series of nonnegative real numbers ("the tail must be small") there is $n \in \mathbb{N}$ such that $\sum_{k=n+1}^\infty \ell(I_k) < \varepsilon/2$. Define $\mathcal{O} = \cup_{k=1}^n I_k$. Since $\mathcal{O} \setminus E \subset \mathcal{U} \setminus E$, then by monotonicity of outer measure (Lemma 2.2.A) and the fact that $m^*(\mathcal{U} \setminus E) < \varepsilon/2$ established above, we have $m^*(\mathcal{O} \setminus E) \leq m^*(\mathcal{U} \setminus E) < \varepsilon/2$. On the other hand, since $E \subset \mathcal{U}$, $E \setminus \mathcal{O} \subset \mathcal{U} \setminus \mathcal{O} = \cup_{k=n+1}^\infty I_k$, and so by the definition of outer measure (in terms of an infimum), $m^*(E \setminus \mathcal{O}) \leq \sum_{k=n+1}^\infty \ell(I_k) < \varepsilon/2$. Thus $m^*(\mathcal{O} \setminus E) + m^*(E \setminus \mathcal{O}) < \varepsilon/2 + \varepsilon/2 = \varepsilon$. □