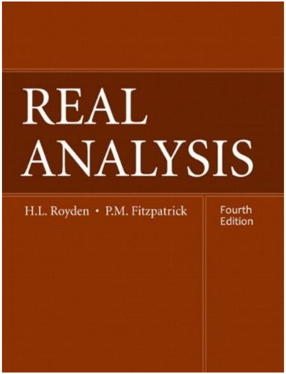
Real Analysis

Chapter 2. Lebesgue Measure

2.4. Outer and Inner Approximation of Lebesgue Measurable Sets—Proofs of Theorems



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If A is measurable and $m^*(A) < \infty$ and $A \subset B$ then

$$m^*(B \setminus A) = m^*(B) - m^*(A).$$

Proof. Since A is measurable and $B \subset \mathbb{R}$ then by the definition of measurable,

$$\begin{aligned} m^*(B) &= m^*(B \cap A) + m^*(B \cap A^c) \\ &= m^*(B \cap A) + m^*(B \setminus A) \text{ since } B \cap A^c = B \setminus A \\ &= m^*(A) + m^*(B \setminus A) \text{ since } A \subset B. \end{aligned}$$

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Theorem 2.11

Theorem 2.11. Let $E \subset \mathbb{R}$. Then each of the following are equivalent to the measurability of E:

(i) For each $\varepsilon > 0$, there is an open set $\mathcal{O}$ containing E for which $m^*(\mathcal{O} \setminus E) < \varepsilon$.

(ii) There is a G_δ set G containing E for which $m^*(G \setminus E) = 0$.

(iii) For each $\varepsilon > 0$, there is a closed set F contained in E for which $m^*(E \setminus F) < \varepsilon$.

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Proof. measurable $\Rightarrow$ (i) Let $E \in \mathcal{M}$ and $\varepsilon > 0$. First, suppose $m^*(E) < \infty$. Then from the definition of outer measure, there is an open cover of intervals $\{I_k\}_{k=1}^{\infty}$ of E for which $\sum \ell(I_k) < m^*(E) + \varepsilon$ (by Theorem 0.3(b)). Define $\mathcal{O} = \cup I_k$. Then $\mathcal{O}$ is open and $E \subset \mathcal{O}$.

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Proof (continued). (i) $\Rightarrow$ (ii) Suppose (i) holds for E. Then for each $k \in \mathbb{N}$ there is open $\mathcal{O}_k \supset E$ where $m^*(\mathcal{O}_k \setminus E) < 1/k$. Define $G = \cap \mathcal{O}_k$. Then G is G_δ and $G \supset E$.

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To show that $E \in \mathcal{M}$ is equivalent to (iii) and (iv) (and hence to (i) and (ii)), we need to apply DeMorgan's Laws and the facts that if $E \in \mathcal{M}$ then $E^c \in \mathcal{M}$, a set is open if and only if its complement is closed, and a set is F_σ if and only if its complement is G_δ (this is Exercise 2.16). □

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$$m^*(E \Delta \mathcal{O}) = m^*(E \setminus \mathcal{O}) + m^*(\mathcal{O} \setminus E) < \varepsilon.$$

Proof. Let $\varepsilon > 0$ be given. Since E is measurable, by Theorem 2.11 (the "measurable implies (i)" part), there is an open set $\mathcal{U}$ such that $E \subset \mathcal{U}$ and $m^*(\mathcal{U} \setminus E) < \varepsilon/2$.

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So $m^*(\mathcal{U}) < \infty$.

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So $m^*(\mathcal{U}) < \infty$. Since $\mathcal{U}$ is an open set of real numbers, then by Theorem 0.7 we have $\mathcal{U} = \cup_{k=1}^\infty I_k$ for some set of open intervals $\{I_k\}_{k=1}^\infty$. Each interval is measurable by Proposition 2.8 and the outer measure of an interval is its length by Proposition 2.1.

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Proof (continued). Therefore by finite additivity (Proposition 2.6) and the monotonicity of outer measure (Lemma 2.2.A) we have for each $n \in \mathbb{N}$:

$$\sum_{k=1}^{n} \ell(I_k) = m^* \left(\cup_{k=1}^{n} I_k \right) \le m^*(\mathcal{U}) < \infty.$$

Therefore $\sum_{k=1}^{\infty} \ell(I_k) < \infty$ and so $\{\ell(I_k)\}_{k=1}^{\infty}$ is a summable sequence of nonnegative real numbers. So by a property of summable series of nonnegative real numbers ("the tail must be small") there is $n \in \mathbb{N}$ such that $\sum_{k=n+1}^{\infty} \ell(I_k) < \varepsilon/2$.

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Therefore $\sum_{k=1}^{\infty} \ell(I_k) < \infty$ and so $\{\ell(I_k)\}_{k=1}^{\infty}$ is a summable sequence of nonnegative real numbers. So by a property of summable series of nonnegative real numbers ("the tail must be small") there is $n \in \mathbb{N}$ such that $\sum_{k=n+1}^{\infty} \ell(I_k) < \varepsilon/2$. Define $\mathcal{O} = \cup_{k=1}^{n} I_k$. Since $\mathcal{O} \setminus E \subset \mathcal{U} \setminus E$, then by monotonicity of outer measure (Lemma 2.2.A) and the fact that $m^*(\mathcal{U} \setminus E) < \varepsilon/2$ established above, we have $m^*(\mathcal{O} \setminus E) \leq m^*(\mathcal{U} \setminus E) < \varepsilon/2$. On the other hand, since $E \subset \mathcal{U}$, $E \setminus \mathcal{O} \subset \mathcal{U} \setminus \mathcal{O} = \cup_{k=n+1}^{\infty} I_k$, and so by the definition of outer measure (in terms of an infimum), $m^*(E \setminus \mathcal{O}) \leq \sum_{k=n+1}^{\infty} \ell(I_k) < \varepsilon/2$. Thus $m^*(\mathcal{O} \setminus E) + m^*(E \setminus \mathcal{O}) < \varepsilon/2 + \varepsilon/2 = \varepsilon$. □