Real Analysis

Chapter 2. Lebesgue Measure

2.5. Countable Additivity, Continuity, and the Borel-Cantelli Lemma—Proofs of Theorems

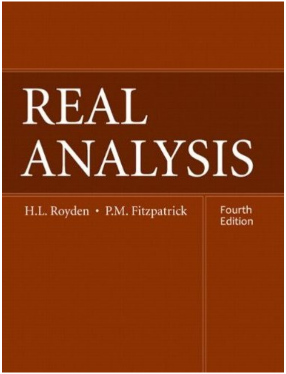

Table of contents

Theorem 2.15

Theorem 2.15. Measure is Continuous.
Lebesgue measure satisfies:

(i) If $\{A_k\}_{k=1}^{\infty}$ is an ascending collection of measurable sets (i.e., $A_k \subset A_{k+1}$), then
$m(\cup_{k=1}^{\infty} A_k) = m(\lim_{k\to\infty} A_k) = \lim_{k\to\infty} m(A_k)$.

(ii) If $\{B_k\}_{k=1}^{\infty}$ is a descending collection of measurable sets (i.e., $B_k \supset B_{k+1}$) and $m(B_1) < \infty$, then
$m(\cap_{k=1}^{\infty} B_k) = m(\lim_{k\to\infty} B_k) = \lim_{k\to\infty} m(B_k)$.

Proof of (i). If $m(A_{k_0}) = \infty$ for some k_0, then the result holds trivially. So suppose, without loss of generality, that $m(A_k) < \infty$ for all k.

Theorem 2.15

Theorem 2.15. Measure is Continuous.
Lebesgue measure satisfies:

(i) If $\{A_k\}_{k=1}^{\infty}$ is an ascending collection of measurable sets (i.e., $A_k \subset A_{k+1}$), then
$m(\cup_{k=1}^{\infty} A_k) = m(\lim_{k\to\infty} A_k) = \lim_{k\to\infty} m(A_k)$.

(ii) If $\{B_k\}_{k=1}^{\infty}$ is a descending collection of measurable sets (i.e., $B_k \supset B_{k+1}$) and $m(B_1) < \infty$, then
$m(\cap_{k=1}^{\infty} B_k) = m(\lim_{k\to\infty} B_k) = \lim_{k\to\infty} m(B_k)$.

Proof of (i). If $m(A_{k_0}) = \infty$ for some k_0, then the result holds trivially. So suppose, without loss of generality, that $m(A_k) < \infty$ for all k. Define $A_0 = \varnothing$ and $C_k = A_k \setminus A_{k-1}$ for $k \geq 1$. Since $\{A_k\}$ is ascending, the C_k's are disjoint and $\cup_{k=1}^{\infty} A_k = \dot\cup_{k=1}^{\infty} C_k$. Since m is countably additive by Proposition 2.6,
$m(\cup_{k=1}^{\infty} A_k) = m(\dot\cup_{k=1}^{\infty} C_k) = \sum_{k=1}^{\infty} m(C_k) = \sum_{k=1}^{\infty} m(A_k \setminus A_{k-1})$.

Theorem 2.15

Theorem 2.15. Measure is Continuous.
Lebesgue measure satisfies:

(i) If $\{A_k\}_{k=1}^{\infty}$ is an ascending collection of measurable sets (i.e., $A_k \subset A_{k+1}$), then
$m(\cup_{k=1}^{\infty} A_k) = m(\lim_{k\to\infty} A_k) = \lim_{k\to\infty} m(A_k)$.

(ii) If $\{B_k\}_{k=1}^{\infty}$ is a descending collection of measurable sets (i.e., $B_k \supset B_{k+1}$) and $m(B_1) < \infty$, then
$m(\cap_{k=1}^{\infty} B_k) = m(\lim_{k\to\infty} B_k) = \lim_{k\to\infty} m(B_k)$.

Proof of (i). If $m(A_{k_0}) = \infty$ for some k_0, then the result holds trivially. So suppose, without loss of generality, that $m(A_k) < \infty$ for all k. Define $A_0 = \varnothing$ and $C_k = A_k \setminus A_{k-1}$ for $k \geq 1$. Since $\{A_k\}$ is ascending, the C_k's are disjoint and $\cup_{k=1}^{\infty} A_k = \dot\cup_{k=1}^{\infty} C_k$. Since m is countably additive by Proposition 2.6,
$m(\cup_{k=1}^{\infty} A_k) = m(\dot\cup_{k=1}^{\infty} C_k) = \sum_{k=1}^{\infty} m(C_k) = \sum_{k=1}^{\infty} m(A_k \setminus A_{k-1})$.

Theorem 2.15 (continued 1)

Theorem 2.15. Measure is Continuous.
Lebesgue measure satisfies:

1. If $\{A_k\}_{k=1}^{\infty}$ is an ascending collection of measurable sets (i.e., $A_k \subset A_{k+1}$), then $m(\cup_{k=1}^{\infty} A_k) = m(\lim_{k\to\infty} A_k) = \lim_{k\to\infty} m(A_k)$.

Proof (continued). By the Excision Property of measure (Lemma 2.4.A),

$$m(\cup_{k=1}^{\infty} A_k) = \sum_{k=1}^{\infty} m(A_k \setminus A_{k-1}) = \sum_{k=1}^{\infty} [m(A_k) - m(A_{k-1})]$$

$$= \lim_{n\to\infty} \left(\sum_{k=1}^{n} [m(A_k) - m(A_{k-1})] \right) = \lim_{n\to\infty} [m(A_n) - m(A_0)] = \lim_{n\to\infty} m(A_n),$$

since $m(A_0) = m(\varnothing) = 0$. Therefore
$m(\cup_{k=1}^{\infty} A_k) = m(\lim_{k\to\infty} A_k) = \lim_{k\to\infty} m(A_k)$, as claimed.

Theorem 2.15 (continued 2)

Theorem 2.15. Measure is Continuous.
Lebesgue measure satisfies:

(ii) If $\{B_k\}_{k=1}^{\infty}$ is a descending collection of measurable sets (i.e., $B_k \supset B_{k+1}$) and $m(B_1) < \infty$, then $m(\cap_{k=1}^{\infty} B_k) = m(\lim_{k\to\infty} B_k) = \lim_{k\to\infty} m(B_k)$.

Proof of (ii). Define $D_k = B_1 \setminus B_k$ for $k \in \mathbb{N}$. Since $\{B_k\}_{k=1}^{\infty}$ is a descending sequence of sets, then $\{D_k\}_{k=1}^{\infty}$ is an ascending sequence of sets. Applying (i) to $\{D_k\}_{k=1}^{\infty}$ gives

$$m\left(\cup_{k=1}^{\infty} D_k\right) = \lim_{k\to\infty} m(D_k). \qquad (*)$$

By De Morgan's Laws (Theorem 0.1, applied to relative complements)

$$\cup_{k=1}^{\infty} D_k = \cup_{k=1}^{\infty}(B_1 \setminus B_k) = \cup_{k=1}^{\infty}(B_1 \cap B_k^c) = B_1 \setminus \cap_{k=1}^{\infty} B_k. \qquad (**)$$

Theorem 2.15 (continued 2)

Theorem 2.15. Measure is Continuous.
Lebesgue measure satisfies:

(ii) If $\{B_k\}_{k=1}^\infty$ is a descending collection of measurable sets (i.e., $B_k \supset B_{k+1}$) and $m(B_1) < \infty$, then $m(\cap_{k=1}^\infty B_k) = m(\lim_{k\to\infty} B_k) = \lim_{k\to\infty} m(B_k)$.

Proof of (ii). Define $D_k = B_1 \setminus B_k$ for $k \in \mathbb{N}$. Since $\{B_k\}_{k=1}^\infty$ is a descending sequence of sets, then $\{D_k\}_{k=1}^\infty$ is an ascending sequence of sets. Applying (i) to $\{D_k\}_{k=1}^\infty$ gives

$$m\left(\cup_{k=1}^\infty D_k\right) = \lim_{k\to\infty} m(D_k). \qquad (*)$$

By De Morgan's Laws (Theorem 0.1, applied to relative complements)

$$\cup_{k=1}^\infty D_k = \cup_{k=1}^\infty (B_1 \setminus B_k) = \cup_{k=1}^\infty (B_1 \cap B_k^c) = B_1 \setminus \cap_{k=1}^\infty B_k. \qquad (**)$$

Theorem 2.15 (continued 3)

Proof (continued). Next, by the Excision Property (Lemma 2.4.A), since $m(B_k) < \infty$ and $B_k \subset B_1$, we have
$m(D_k) = m(B_1 \setminus B_k) = m(B_1) - m(B_k)$ for all $k \in \mathbb{N}$. So

$$\begin{aligned} m\left(\cup_{k=1}^{\infty} D_k\right) &= m\left(B_1 \setminus \cap_{k=1}^{\infty} B_k\right) \text{ by } (**) \\ &= m(B_1) - m\left(\cap_{k=1}^{\infty} B_k\right) \text{ by the Excision Property} \\ &= \lim_{k\to\infty} m(D_k) \text{ by } (*) \\ &= \lim_{k\to\infty} (m(B_1) - m(B_k)) \text{ by the definition of } D_k \\ &= m(B_1) - \lim_{k\to\infty} m(B_k). \end{aligned}$$

Hence, since $m(B_1) < \infty$, $m\left(\cap_{k=1}^{\infty} B_k\right) = \lim_{k\to\infty} m(B_k)$, as claimed. □

Theorem 2.15 (continued 3)

Proof (continued). Next, by the Excision Property (Lemma 2.4.A), since $m(B_k) < \infty$ and $B_k \subset B_1$, we have $m(D_k) = m(B_1 \setminus B_k) = m(B_1) - m(B_k)$ for all $k \in \mathbb{N}$. So

$$\begin{aligned} m\left(\cup_{k=1}^{\infty} D_k\right) &= m\left(B_1 \setminus \cap_{k=1}^{\infty} B_k\right) \text{ by } (**) \\ &= m(B_1) - m\left(\cap_{k=1}^{\infty} B_k\right) \text{ by the Excision Property} \\ &= \lim_{k\to\infty} m(D_k) \text{ by } (*) \\ &= \lim_{k\to\infty} (m(B_1) - m(B_k)) \text{ by the definition of } D_k \\ &= m(B_1) - \lim_{k\to\infty} m(B_k). \end{aligned}$$

Hence, since $m(B_1) < \infty$, $m\left(\cap_{k=1}^{\infty} B_k\right) = \lim_{k\to\infty} m(B_k)$, as claimed. □

The Borel-Cantelli Lemma

The Borel-Cantelli Lemma.
Let $\{E_k\}_{k=1}^{\infty}$ be a countable collection of measurable sets for which $\sum_{k=1}^{\infty} m(E_k) < \infty$. Then almost all $x \in \mathbb{R}$ belong to at most finitely many of the E_k's.

Proof. By countable subadditivity $m(\cup_{k=n}^{\infty} E_k) \leq \sum_{k=n}^{\infty} m(E_k) < \infty$. So

$$\begin{aligned} m(\cap_{n=1}^{\infty}[\cup_{k=n}^{\infty} E_k]) &= \lim_{n\to\infty} m(\cup_{k=n}^{\infty} E_k) \text{ by Theorem 2.15(ii)} \\ &\leq \lim_{n\to\infty} \sum_{k=n}^{\infty} m(E_k) \text{ as above} \\ &= 0 \text{ since } \sum_{n=1}^{\infty} m(E_k) < \infty. \end{aligned}$$

The Borel-Cantelli Lemma

The Borel-Cantelli Lemma.
Let $\{E_k\}_{k=1}^{\infty}$ be a countable collection of measurable sets for which $\sum_{k=1}^{\infty} m(E_k) < \infty$. Then almost all $x \in \mathbb{R}$ belong to at most finitely many of the E_k's.

Proof. By countable subadditivity $m(\cup_{k=n}^{\infty} E_k) \leq \sum_{k=n}^{\infty} m(E_k) < \infty$. So

$$\begin{aligned} m(\cap_{n=1}^{\infty}[\cup_{k=n}^{\infty} E_k]) &= \lim_{n\to\infty} m(\cup_{k=n}^{\infty} E_k) \text{ by Theorem 2.15(ii)} \\ &\leq \lim_{n\to\infty} \sum_{k=n}^{\infty} m(E_k) \text{ as above} \\ &= 0 \text{ since } \sum_{n=1}^{\infty} m(E_k) < \infty. \end{aligned}$$

Now $\cap_{n=1}^{\infty}[\cup_{k=n}^{\infty} E_k]$ is the set of all points which are in infinitely many E_k's. Since the measure of this set is zero, almost all real numbers belong to finitely many E_k's, as claimed. □

The Borel-Cantelli Lemma

The Borel-Cantelli Lemma.
Let $\{E_k\}_{k=1}^{\infty}$ be a countable collection of measurable sets for which $\sum_{k=1}^{\infty} m(E_k) < \infty$. Then almost all $x \in \mathbb{R}$ belong to at most finitely many of the E_k's.

Proof. By countable subadditivity $m(\cup_{k=n}^{\infty} E_k) \leq \sum_{k=n}^{\infty} m(E_k) < \infty$. So

$$\begin{aligned} m(\cap_{n=1}^{\infty}[\cup_{k=n}^{\infty} E_k]) &= \lim_{n\to\infty} m(\cup_{k=n}^{\infty} E_k) \text{ by Theorem 2.15(ii)} \\ &\leq \lim_{n\to\infty} \sum_{k=n}^{\infty} m(E_k) \text{ as above} \\ &= 0 \text{ since } \sum_{n=1}^{\infty} m(E_k) < \infty. \end{aligned}$$

Now $\cap_{n=1}^{\infty}[\cup_{k=n}^{\infty} E_k]$ is the set of all points which are in infinitely many E_k's. Since the measure of this set is zero, almost all real numbers belong to finitely many E_k's, as claimed. □