Real Analysis

Chapter 2. Lebesgue Measure

2.6. Nonmeasurable Sets (4th Ed.)—Proofs of Theorems

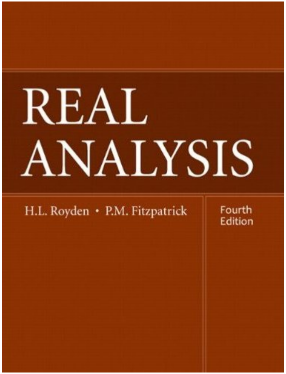

Table of contents

Lemma 2.16

Lemma 2.16. Let E be a bounded measurable set of real numbers. Suppose there is a bounded countably infinite set of real numbers Λ for which the collection of translates of E, $\{E + \lambda\}_{\lambda \in \Lambda}$ is disjoint. Then $m(E) = 0$.

Proof. The translate of a measurable set is measurable by Proposition 2.10. So by countable additivity (Proposition 2.13) $m\left(\dot{\cup}_{\lambda \in \Lambda}(E + \lambda)\right) = \sum_{\lambda \in \Lambda} m(E + \lambda)$.

Lemma 2.16

Lemma 2.16. Let E be a bounded measurable set of real numbers. Suppose there is a bounded countably infinite set of real numbers Λ for which the collection of translates of E, $\{E + \lambda\}_{\lambda \in \Lambda}$ is disjoint. Then $m(E) = 0$.

Proof. The translate of a measurable set is measurable by Proposition 2.10. So by countable additivity (Proposition 2.13) $m\left(\dot\cup_{\lambda \in \Lambda}(E + \lambda)\right) = \sum_{\lambda \in \Lambda} m(E + \lambda)$. Since both E and Λ are bounded sets, then the set $\dot\cup_{\lambda \in \Lambda}(E + \lambda)$ is also bounded (below by a sum of lower bounds of E and Λ and above by a sum of upper bounds of E and Λ) and so (by monotonicity, Lemma 2.2.A) has finite measure. Since measure is translation invariant (Proposition 2.2), $m(E + \lambda) = m(E) \geq 0$.

Lemma 2.16

Lemma 2.16. Let E be a bounded measurable set of real numbers. Suppose there is a bounded countably infinite set of real numbers Λ for which the collection of translates of E, $\{E+\lambda\}_{\lambda\in\Lambda}$ is disjoint. Then $m(E)=0$.

Proof. The translate of a measurable set is measurable by Proposition 2.10. So by countable additivity (Proposition 2.13) $m\left(\dot\cup_{\lambda\in\Lambda}(E+\lambda)\right)=\sum_{\lambda\in\Lambda} m(E+\lambda)$. Since both E and Λ are bounded sets, then the set $\dot\cup_{\lambda\in\Lambda}(E+\lambda)$ is also bounded (below by a sum of lower bounds of E and Λ and above by a sum of upper bounds of E and Λ) and so (by monotonicity, Lemma 2.2.A) has finite measure. Since measure is translation invariant (Proposition 2.2), $m(E+\lambda)=m(E)\geq 0$. Then

$$\infty > m\left(\dot\cup_{\lambda\in\Lambda}(E+\lambda)\right)=\sum_{\lambda\in\Lambda} m(E+\lambda)=\sum_{\lambda\in\Lambda} m(E)$$

and so $m(E)=0$ (otherwise, $\sum_{\lambda\in\Lambda} m(E)=\infty$). □

Lemma 2.16

Lemma 2.16. Let E be a bounded measurable set of real numbers. Suppose there is a bounded countably infinite set of real numbers Λ for which the collection of translates of E, $\{E+\lambda\}_{\lambda\in\Lambda}$ is disjoint. Then $m(E)=0$.

Proof. The translate of a measurable set is measurable by Proposition 2.10. So by countable additivity (Proposition 2.13) $m\left(\dot\cup_{\lambda\in\Lambda}(E+\lambda)\right)=\sum_{\lambda\in\Lambda}m(E+\lambda)$. Since both E and Λ are bounded sets, then the set $\dot\cup_{\lambda\in\Lambda}(E+\lambda)$ is also bounded (below by a sum of lower bounds of E and Λ and above by a sum of upper bounds of E and Λ) and so (by monotonicity, Lemma 2.2.A) has finite measure. Since measure is translation invariant (Proposition 2.2), $m(E+\lambda)=m(E)\geq 0$. Then

$$\infty > m\left(\dot\cup_{\lambda\in\Lambda}(E+\lambda)\right)=\sum_{\lambda\in\Lambda}m(E+\lambda)=\sum_{\lambda\in\Lambda}m(E)$$

and so $m(E)=0$ (otherwise, $\sum_{\lambda\in\Lambda}m(E)=\infty$). □

Theorem 2.17

Theorem 2.17. The Vitali Construction of a Nonmeasurable Set. Any set E of real numbers with positive outer measure contains a subset that fails to be measurable.

Proof. By Exercise 2.14, E has a bounded subset of positive outer measure, so without loss of generality we may suppose that E is bounded. Let $\mathcal{C}_E$ be a choice set for the rational equivalence relation on E. We now show $\mathcal{C}_E$ is not measurable.

Theorem 2.17

Theorem 2.17. The Vitali Construction of a Nonmeasurable Set. Any set E of real numbers with positive outer measure contains a subset that fails to be measurable.

Proof. By Exercise 2.14, E has a bounded subset of positive outer measure, so without loss of generality we may suppose that E is bounded. Let $\mathcal{C}_E$ be a choice set for the rational equivalence relation on E. We now show $\mathcal{C}_E$ is not measurable.

ASSUME $\mathcal{C}_E$ is measurable. Let Λ_0 be *any* bounded countably infinite set of rational numbers. Since $\mathcal{C}_E$ is measurable and collection $\{\mathcal{C}_E + \lambda\}_{\lambda \in \Lambda_0}$ is disjoint, then by Lemma 2.16 we have $m(\mathcal{C}_E) = 0$.

Theorem 2.17

Theorem 2.17. The Vitali Construction of a Nonmeasurable Set. Any set E of real numbers with positive outer measure contains a subset that fails to be measurable.

Proof. By Exercise 2.14, E has a bounded subset of positive outer measure, so without loss of generality we may suppose that E is bounded. Let $\mathcal{C}_E$ be a choice set for the rational equivalence relation on E. We now show $\mathcal{C}_E$ is not measurable.

ASSUME $\mathcal{C}_E$ is measurable. Let Λ_0 be *any* bounded countably infinite set of rational numbers. Since $\mathcal{C}_E$ is measurable and collection $\{\mathcal{C}_E + \lambda\}_{\lambda \in \Lambda_0}$ is disjoint, then by Lemma 2.16 we have $m(\mathcal{C}_E) = 0$. So by the translation invariance (Proposition 2.2) and countable additivity (Proposition 2.13),

$$m\left(\cup_{\lambda \in \Lambda_0} (\mathcal{C}_E + \lambda)\right) = \sum_{\lambda \in \Lambda_0} m(\mathcal{C}_E + \lambda) = \sum_{\lambda \in \Lambda_0} m(\mathcal{C}_E) = 0.$$

Theorem 2.17

Theorem 2.17. The Vitali Construction of a Nonmeasurable Set. Any set E of real numbers with positive outer measure contains a subset that fails to be measurable.

Proof. By Exercise 2.14, E has a bounded subset of positive outer measure, so without loss of generality we may suppose that E is bounded. Let $\mathcal{C}_E$ be a choice set for the rational equivalence relation on E. We now show $\mathcal{C}_E$ is not measurable.

ASSUME $\mathcal{C}_E$ is measurable. Let Λ_0 be *any* bounded countably infinite set of rational numbers. Since $\mathcal{C}_E$ is measurable and collection $\{\mathcal{C}_E + \lambda\}_{\lambda \in \Lambda_0}$ is disjoint, then by Lemma 2.16 we have $m(\mathcal{C}_E) = 0$. So by the translation invariance (Proposition 2.2) and countable additivity (Proposition 2.13),

$$m\left(\dot{\cup}_{\lambda \in \Lambda_0}(\mathcal{C}_E + \lambda)\right) = \sum_{\lambda \in \Lambda_0} m(\mathcal{C}_E + \lambda) = \sum_{\lambda \in \Lambda_0} m(\mathcal{C}_E) = 0.$$

Theorem 2.17 (continued)

Theorem 2.17. The Vitali Construction of a Nonmeasurable Set.
Any set E of real numbers with positive outer measure contains a subset that fails to be measurable.

Proof (continued). Since E is bounded, then $E \subset [-b, b]$ for some $b \in \mathbb{R}$. Define $\Lambda_0 = [-2b, 2b] \cap \mathbb{Q}$. Then Λ_0 is bounded and countably infinite. Let $x \in E$. Then there is some $c \in \mathcal{C}_E$ such that $x = c + q$ with $q \in \mathbb{Q}$. But x and c belong to $[-b, b]$ and so $x = c + q \in [-b, b]$. So $x = c + q \in \mathcal{C}_E + \lambda$ where $\lambda = q \in [-2b, 2b]$. Since x is an arbitrary element of E then $E \subset \cup_{\lambda \in \Lambda_0}(\mathcal{C}_E + \lambda)$.

Theorem 2.17 (continued)

Theorem 2.17. The Vitali Construction of a Nonmeasurable Set. Any set E of real numbers with positive outer measure contains a subset that fails to be measurable.

Proof (continued). Since E is bounded, then $E \subset [-b, b]$ for some $b \in \mathbb{R}$. Define $\Lambda_0 = [-2b, 2b] \cap \mathbb{Q}$. Then Λ_0 is bounded and countably infinite. Let $x \in E$. Then there is some $c \in \mathcal{C}_E$ such that $x = c + q$ with $q \in \mathbb{Q}$. But x and c belong to $[-b, b]$ and so $x = c + q \in [-b, b]$. So $x = c + q \in \mathcal{C}_E + \lambda$ where $\lambda = q \in [-2b, 2b]$. Since x is an arbitrary element of E then $E \subset \cup_{\lambda \in \Lambda_0} (\mathcal{C}_E + \lambda)$. But $m^*(E) > 0$ and so as shown above $m^*(\cup_{\lambda \in \Lambda_0}(\mathcal{C}_E + \lambda)) = 0$, so we have a CONTRADICTION by monotonicity (Lemma 2.2.A):

$$0 < m^*(E) \leq m^*(\cup_{\lambda \in \Lambda_0}(\mathcal{C}_E + \lambda)) = 0.$$

This contradiction shows that the assumption that $\mathcal{C}_E$ is measurable is false. □

Theorem 2.17 (continued)

Theorem 2.17. The Vitali Construction of a Nonmeasurable Set. Any set E of real numbers with positive outer measure contains a subset that fails to be measurable.

Proof (continued). Since E is bounded, then $E \subset [-b, b]$ for some $b \in \mathbb{R}$. Define $\Lambda_0 = [-2b, 2b] \cap \mathbb{Q}$. Then Λ_0 is bounded and countably infinite. Let $x \in E$. Then there is some $c \in \mathcal{C}_E$ such that $x = c + q$ with $q \in \mathbb{Q}$. But x and c belong to $[-b, b]$ and so $x = c + q \in [-b, b]$. So $x = c + q \in \mathcal{C}_E + \lambda$ where $\lambda = q \in [-2b, 2b]$. Since x is an arbitrary element of E then $E \subset \cup_{\lambda \in \Lambda_0} (\mathcal{C}_E + \lambda)$. But $m^*(E) > 0$ and so as shown above $m^*(\cup_{\lambda \in \Lambda_0} (\mathcal{C}_E + \lambda)) = 0$, so we have a CONTRADICTION by monotonicity (Lemma 2.2.A):

$$0 < m^*(E) \leq m^*(\cup_{\lambda \in \Lambda_0} (\mathcal{C}_E + \lambda)) = 0.$$

This contradiction shows that the assumption that $\mathcal{C}_E$ is measurable is false. □

Theorem 2.18

Theorem 2.18. There are disjoint sets of real numbers A and B for which

$$m^*(A \mathbin{\dot\cup} B) < m^*(A) + m^*(B).$$

Proof. ASSUME $m^*(A \mathbin{\dot\cup} B) = m^*(A) + m^*(B)$ for every disjoint pair of sets A and B.

Theorem 2.18

Theorem 2.18. There are disjoint sets of real numbers A and B for which

$$m^*(A \cup B) < m^*(A) + m^*(B).$$

Proof. ASSUME $m^*(A \cup B) = m^*(A) + m^*(B)$ for every disjoint pair of sets A and B. Then for any $A, E \subset \mathbb{R}$ we have

$$m^*(A) = m^*((A \cap E) \cup (A \cap E^c)) = m^*(A \cap E) + m^*(A \cap E^c)$$

and so every $E \subset \mathbb{R}$ is measurable, a CONTRADICTION to Theorem 2.17.

Theorem 2.18

Theorem 2.18. There are disjoint sets of real numbers A and B for which

$$m^*(A \cup B) < m^*(A) + m^*(B).$$

Proof. ASSUME $m^*(A \cup B) = m^*(A) + m^*(B)$ for every disjoint pair of sets A and B. Then for any $A, E \subset \mathbb{R}$ we have

$$m^*(A) = m^*((A \cap E) \cup (A \cap E^c)) = m^*(A \cap E) + m^*(A \cap E^c)$$

and so every $E \subset \mathbb{R}$ is measurable, a CONTRADICTION to Theorem 2.17. So for some disjoint $A, B \subset \mathbb{R}$ we have $m^*(A \cup B) \neq m^*(A) + m^*(B)$. By subadditivity (Proposition 2.3) $m^*(A \cup B) \leq m^*(A) + m^*(B)$, so it must be that for some disjoint $A, B \subset \mathbb{R}$ we have $m^*(A \cup B) < m^*(A) + m^*(B)$. □

Theorem 2.18

Theorem 2.18. There are disjoint sets of real numbers A and B for which

$$m^*(A \cup B) < m^*(A) + m^*(B).$$

Proof. ASSUME $m^*(A \cup B) = m^*(A) + m^*(B)$ for every disjoint pair of sets A and B. Then for any $A, E \subset \mathbb{R}$ we have

$$m^*(A) = m^*((A \cap E) \cup (A \cap E^c)) = m^*(A \cap E) + m^*(A \cap E^c)$$

and so every $E \subset \mathbb{R}$ is measurable, a CONTRADICTION to Theorem 2.17. So for some disjoint $A, B \subset \mathbb{R}$ we have $m^*(A \cup B) \neq m^*(A) + m^*(B)$. By subadditivity (Proposition 2.3) $m^*(A \cup B) \leq m^*(A) + m^*(B)$, so it must be that for some disjoint $A, B \subset \mathbb{R}$ we have
$m^*(A \cup B) < m^*(A) + m^*(B)$. □