Chapter 2. Lebesgue Measure
2.6. Nonmeasurable Sets (4th Ed.)—Proofs of Theorems
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Lemma 2.16

Lemma 2.16. Let $E$ be a bounded measurable set of real numbers. Suppose there is a bounded countably infinite set of real numbers $\Lambda$ for which the collection of translates of $E$, $\{E + \lambda\}_{\lambda \in \Lambda}$ is disjoint. Then $m(E) = 0$.

Proof. The translate of a measurable set is measurable by Proposition 2.10. So by countable additivity (Proposition 2.13)

$$m\left(\bigcup_{\lambda \in \Lambda}(E + \lambda)\right) = \sum_{\lambda \in \Lambda} m(E + \lambda).$$
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$m(\bigcup_{\lambda \in \Lambda}(E + \lambda)) = \sum_{\lambda \in \Lambda} m(E + \lambda)$. Since both $E$ and $\Lambda$ are bounded sets, then the set $\bigcup_{\lambda \in \Lambda}(E + \lambda)$ is also bounded (below by a sum of lower bounds of $E$ and $\Lambda$ and above by a sum of upper bounds of $E$ and $\Lambda$) and so (by monotonicity, say) has finite measure. Since measure is translation invariant (Proposition 2.2), $m(E + \lambda) = m(E) \geq 0$. 
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Proof. The translate of a measurable set is measurable by Proposition 2.10. So by countable additivity (Proposition 2.13) $m \left( \bigcup_{\lambda \in \Lambda} (E + \lambda) \right) = \sum_{\lambda \in \Lambda} m(E + \lambda)$. Since both $E$ and $\Lambda$ are bounded sets, then the set $\bigcup_{\lambda \in \Lambda} (E + \lambda)$ is also bounded (below by a sum of lower bounds of $E$ and $\Lambda$ and above by a sum of upper bounds of $E$ and $\Lambda$) and so (by monotonicity, say) has finite measure. Since measure is translation invariant (Proposition 2.2), $m(E + \lambda) = m(E) \geq 0$. Then

$$\infty > m \left( \bigcup_{\lambda \in \Lambda} (E + \lambda) \right) = \sum_{\lambda \in \Lambda} m(E + \lambda) = \sum_{\lambda \in \Lambda} m(E)$$

and so $m(E) = 0$ (otherwise, $\sum_{\lambda \in \Lambda} m(E) = \infty$).
Lemma 2.16

**Lemma 2.16.** Let $E$ be a **bounded** measurable set of real numbers. Suppose there is a **bounded** countably infinite set of real numbers $\Lambda$ for which the collection of translates of $E$, $\{E + \lambda\}_{\lambda \in \Lambda}$ is disjoint. Then $m(E) = 0$.

**Proof.** The translate of a measurable set is measurable by Proposition 2.10. So by countable additivity (Proposition 2.13) $m(\bigcup_{\lambda \in \Lambda} (E + \lambda)) = \sum_{\lambda \in \Lambda} m(E + \lambda)$. Since both $E$ and $\Lambda$ are bounded sets, then the set $\bigcup_{\lambda \in \Lambda} (E + \lambda)$ is also bounded (below by a sum of lower bounds of $E$ and $\Lambda$ and above by a sum of upper bounds of $E$ and $\Lambda$) and so (by monotonicity, say) has finite measure. Since measure is translation invariant (Proposition 2.2), $m(E + \lambda) = m(E) \geq 0$. Then

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Theorem 2.17. The Vitali Construction of a Nonmeasurable Set.

Any set $E$ of real numbers with positive outer measure contains a subset that fails to be measurable.

Proof. By Exercise 2.14, $E$ has a bounded subset of positive outer measure, so without loss of generality we may suppose that $E$ is bounded. Let $C_E$ be a choice set for the rational equivalence relation on $E$. We now show $C_E$ is not measurable.
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ASSUME $C_E$ is measurable. Let $\Lambda_0$ be any bounded countably infinite set of rational numbers. Since $C_E$ is measurable and collection $\{C_E + \lambda\}_{\lambda \in \Lambda_0}$ is disjoint, then by Lemma 2.16 we have $m(C_E) = 0$. 

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Assume $C_E$ is measurable. Let $\Lambda_0$ be any bounded countably infinite set of rational numbers. Since $C_E$ is measurable and collection $\{C_E + \lambda\}_{\lambda \in \Lambda_0}$ is disjoint, then by Lemma 2.16 we have $m(C_E) = 0$. So by the translation invariance (Proposition 2.2) and countable additivity (Proposition 2.13),

$$m\left(\bigcup_{\lambda \in \Lambda_0} (C_E + \lambda)\right) = \sum_{\lambda \in \Lambda_0} m(C_E + \lambda) = \sum_{\lambda \in \Lambda_0} m(C_E) = 0.$$
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Assume $C_E$ is measurable. Let $\Lambda_0$ be any bounded countably infinite set of rational numbers. Since $C_E$ is measurable and collection $\{C_E + \lambda\}_{\lambda \in \Lambda_0}$ is disjoint, then by Lemma 2.16 we have $m(C_E) = 0$. So by the translation invariance (Proposition 2.2) and countable additivity (Proposition 2.13),

$$m\left(\bigcup_{\lambda \in \Lambda_0} (C_E + \lambda)\right) = \sum_{\lambda \in \Lambda_0} m(C_E + \lambda) = \sum_{\lambda \in \Lambda_0} m(C_E) = 0.$$
Theorem 2.17 (continued)

Theorem 2.17. The Vitali Construction of a Nonmeasurable Set.
Any set $E$ of real numbers with positive outer measure contains a subset that fails to be measurable.

Proof (continued). Since $E$ is bounded, then $E \subset [-b, b]$ for some $b \in \mathbb{R}$. Let $\Lambda_0 = [-2b, 2b] \cap \mathbb{Q}$. Then $\Lambda_0$ is bounded and countably infinite. Let $x \in E$. Then there is some $c \in C_E$ such that $x = c + q$ with $q \in \mathbb{Q}$. But $x$ and $c$ belong to $[-b, b]$ and so $x = c + q \in [-2b, 2b]$. So $x = c + q \in C_E + \lambda$ where $\lambda = q \in [-2b, 2b]$. Since $x$ is an arbitrary element of $E$ then $E \subset \bigcup_{\lambda \in \Lambda_0} (C_E + \lambda)$. 
Theorem 2.17 (continued)

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$$0 < m^*(E) = m^*(\cup_{\lambda \in \Lambda_0} (C_E + \lambda)) = 0.$$ 

This contradiction shows that the assumption that $C_E$ is measurable if false.
Theorem 2.17. The Vitali Construction of a Nonmeasurable Set.

Any set \( E \) of real numbers with positive outer measure contains a subset that fails to be measurable.

**Proof (continued).** Since \( E \) is bounded, then \( E \subset [-b, b] \) for some \( b \in \mathbb{R} \). Let \( \Lambda_0 = [-2b, 2b] \cap \mathbb{Q} \). Then \( \Lambda_0 \) is bounded and countably infinite. Let \( x \in E \). Then there is some \( c \in \mathcal{C}_E \) such that \( x = c + q \) with \( q \in \mathbb{Q} \). But \( x \) and \( c \) belong to \( [-b, b] \) and so \( x = c + q \in [-2b, 2b] \). So \( x = c + q \in \mathcal{C}_E + \lambda \) where \( \lambda = q \in [-2b, 2b] \). Since \( x \) is an arbitrary element of \( E \) then \( E \subset \bigcup_{\lambda \in \Lambda_0} (\mathcal{C}_E + \lambda) \). But \( m^*(E) > 0 \) and so as shown above \( m^*(\bigcup_{\lambda \in \Lambda_0} (\mathcal{C}_E + \lambda)) = 0 \), so we have a CONTRADICTION by monotonicity (Lemma 2.2.A):

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0 < m^*(E) = m^*(\bigcup_{\lambda \in \Lambda_0} (\mathcal{C}_E + \lambda)) = 0.
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This contradiction shows that the assumption that \( \mathcal{C}_E \) is measurable if false.
Theorem 2.18. There are disjoint sets of real numbers $A$ and $B$ for which

$$m^*(A \cup B) < m^*(A) + m^*(B).$$

Proof. ASSUME $m^*(A \cup B) = m^*(A) + m^*(B)$ for every disjoint pair of sets $A$ and $B$. 
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Proof. ASSUME $m^*(A \cup B) = m^*(A) + m^*(B)$ for every disjoint pair of sets $A$ and $B$. Then for any $A, E \subset \mathbb{R}$ we have

$$m^*(A) = m^*((A \cap E) \cup (A \cap E^c)) = m^*(A \cap E) + m^*(A \cap E^c)$$

and so every $E \subset \mathbb{R}$ is measurable, a CONTRADICTION to Theorem 2.17.
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and so every $E \subset \mathbb{R}$ is measurable, a CONTRADICTION to Theorem 2.17. So for some disjoint $A, B \subset \mathbb{R}$ we have $m^*(A \cup B) \neq m^*(A) + m^*(B)$. By subadditivity (Proposition 2.3) $m^*(A \cup B) \leq m^*(A) + m^*(B)$, so it must be that for some disjoint $A, B \subset \mathbb{R}$ we have $m^*(A \cup B) < m^*(A) + m^*(B)$. □
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