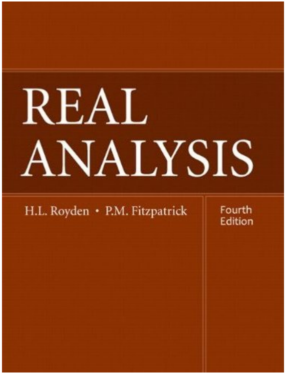
Real Analysis

Chapter 2. Lebesgue Measure

2.6. Nonmeasurable Sets (3rd Ed.)—Proofs of Theorems



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Lemma 2.6.A

Lemma 2.6.A. Let $E \subset [0,1)$ and $E \in \mathcal{M}$. Then for all $y \in [0,1)$, $E \mathring{+} y$ is measurable and $m(E \mathring{+} y) = m(E)$.

Proof. Define $E_1 = E \cap [0, 1-y)$ and $E_2 = E \cap [1-y, 1)$. Then $E_1 \cap E_2 = \varnothing$, $E = E_1 \cup E_2$, and $E_1, E_2 \in \mathcal{M}$. So $m(E) = m(E_1) + m(E_2)$ by countable additivity (Proposition 2.13).

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Proof. Define $E_1 = E \cap [0, 1-y)$ and $E_2 = E \cap [1-y, 1)$. Then $E_1 \cap E_2 = \varnothing$, $E = E_1 \mathbin{\dot\cup} E_2$, and $E_1, E_2 \in \mathcal{M}$. So $m(E) = m(E_1) + m(E_2)$ by countable additivity (Proposition 2.13). Now $E_1 \dotplus y = E_1 + y$ and so $E_1 \dotplus y \in \mathcal{M}$ and $m(E_1 \dotplus y) = m(E_1)$ since m is translation invariant (Proposition 2.2). Also, $E_2 \dotplus y = (E_2 + y) - 1 = E_2 + (y-1)$ and so $E_2 \dotplus y \in \mathcal{M}$ and $m(E_2 \dotplus y) = m(E_2)$. Next, $E \dotplus y = (E_1 \dotplus y) \mathbin{\dot\cup} (E_2 \dotplus y)$, so $E \dotplus y \in \mathcal{M}$ and so by countable additivity (Proposition 2.13):

$$m(E \dotplus y) = m(E_1 \dotplus y) + m(E_2 \dotplus y) = m(E_1) + m(E_2) = m(E).$$

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Proof. Define $E_1 = E \cap [0, 1-y)$ and $E_2 = E \cap [1-y, 1)$. Then $E_1 \cap E_2 = \varnothing$, $E = E_1 \,\dot{\cup}\, E_2$, and $E_1, E_2 \in \mathcal{M}$. So $m(E) = m(E_1) + m(E_2)$ by countable additivity (Proposition 2.13). Now $E_1 \dot{+} y = E_1 + y$ and so $E_1 \dot{+} y \in \mathcal{M}$ and $m(E_1 \dot{+} y) = m(E_1)$ since m is translation invariant (Proposition 2.2). Also, $E_2 \dot{+} y = (E_2 + y) - 1 = E_2 + (y-1)$ and so $E_2 \dot{+} y \in \mathcal{M}$ and $m(E_2 \dot{+} y) = m(E_2)$. Next, $E \dot{+} y = (E_1 \dot{+} y) \,\dot{\cup}\, (E_2 \dot{+} y)$, so $E \dot{+} y \in \mathcal{M}$ and so by countable additivity (Proposition 2.13):

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Theorem 2.6.B

Theorem 2.6.B. Set P is not measurable.

Proof. First, we establish some set theoretic results. Let $\{r_i\}_{i=0}^{\infty}$ be an enumeration of $\mathbb{Q} \cap [0,1)$ where $r_0 = 0$. Define $P_i = P \dot{+} r_i$. Then $P_0 = P$.

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If $x \in P_i \cap P_j$, then $x = p_i \mathring{+} r_i = p_j \mathring{+} r_j$ where $p_i, p_j \in P$. But then $p_i \mathring{+} (-p_j) = r_j \mathring{+} (-r_i) \in \mathbb{Q}$ and so $p_i \sim p_j$. So p_i and p_j are from the same equivalence class under $\sim$ and since P contains only one representative from each equivalence class, then $p_i = p_j$ and $P_i = P_j$. Therefore $P_i \cap P_j = \varnothing$ if $i \neq j$ and so the P_i's are disjoint and $\cup_{i=1}^{\infty} P_i \subset [0,1)$.

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Let $x \in [0, 1)$. Then x is in some equivalence class E_x. Let $p_x \in P$ be the representative of class E_x (i.e., $f(E_x) = p_x$ for choice function f).

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Theorem 2.6.B (continued)

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Proof (continued). ASSUME P is measurable. Then by Lemma 2.6.A, each P_i is measurable and $m(P_i) = m(P)$.

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$$\begin{aligned} 1 &= m([0,1)) \text{ by Propositions 2.1 and 2.8} \\ &= m(\uplus_{i=1}^{\infty} P_i) \text{ since } [0,1) = \uplus_{i=1}^{\infty} P_i \\ &= \sum_{i=1}^{\infty} m(P_i) \text{ by countable additivity (Proposition 2.13)} \\ &= \sum_{i=1}^{\infty} m(P) \text{ since } m(P) = m(P_i) \text{ for all } i \in \mathbb{N} \cup \{0\} \\ &= \begin{cases} 0 & \text{if } m(P) = 0 \\ \infty & \text{if } m(P) > 0, \end{cases} \end{aligned}$$

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There are disjoint sets of real numbers A and B for which

$$m^*(A \cup B) < m^*(A) + m^*(B).$$

Proof. ASSUME $m^*(A \cup B) = m^*(A) + m^*(B)$ for every disjoint pair of sets A and B.

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Proof. ASSUME $m^*(A \cup B) = m^*(A) + m^*(B)$ for every disjoint pair of sets A and B. Then for any $A, E \subset \mathbb{R}$ we have

$$m^*(A) = m^*((A \cap E) \cup (A \cap E^c)) = m^*(A \cap E) + m^*(A \cap E^c)$$

and so every $E \subset \mathbb{R}$ is measurable, a CONTRADICTION to Corollary 2.6.C.

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and so every $E \subset \mathbb{R}$ is measurable, a CONTRADICTION to Corollary 2.6.C. So for some disjoint $A, B \subset \mathbb{R}$ we have $m^*(A \cup B) \neq m^*(A) + m^*(B)$. By subadditivity (Proposition 2.3) $m^*(A \cup B) \leq m^*(A) + m^*(B)$, so it must be that for some disjoint $A, B \subset \mathbb{R}$ we have $m^*(A \cup B) < m^*(A) + m^*(B)$. □

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