Chapter 2. Lebesgue Measure
2.6. Nonmeasurable Sets (3rd Ed.)—Proofs of Theorems
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Lemma 2.6.A. Let $E \subset [0, 1)$ and $E \in \mathcal{M}$. Then for all $y \in [0, 1)$, $E \mathbin{\hat{+}} y$ is measurable and $m(E \mathbin{\hat{+}} y) = m(E)$.

Proof. Define $E_1 = E \cap [0, 1 - y)$ and $E_2 = E \cap [1 - y, 1)$. Then $E_1 \cap E_2 = \emptyset$, $E = E_1 \cup E_2$, and $E_1, E_2 \in \mathcal{M}$. So $m(E) = m(E_1) + m(E_2)$ by countable additivity (Proposition 2.13).
Lemma 2.6.A. Let $E \subset [0, 1)$ and $E \in \mathcal{M}$. Then for all $y \in [0, 1)$, $E + y$ is measurable and $m(E + y) = m(E)$.

Proof. Define $E_1 = E \cap [0, 1 - y)$ and $E_2 = E \cap [1 - y, 1)$. Then $E_1 \cap E_2 = \emptyset$, $E = E_1 \cup E_2$, and $E_1, E_2 \in \mathcal{M}$. So $m(E) = m(E_1) + m(E_2)$ by countable additivity (Proposition 2.13). Now $E_1 + y = E_1 + y$ and so $E_1 + y \in \mathcal{M}$ and $m(E_1 + y) = m(E)$ since $m$ is translation invariant (Proposition 2.2). Also, $E_2 + y = (E_2 + y) - 1 = E_2 + (y - 1)$ and so $E_2 + y \in \mathcal{M}$ and $m(E_2 + y) = m(E_2)$. 
Lemma 2.6.A. Let $E \subset [0, 1)$ and $E \in \mathcal{M}$. Then for all $y \in [0, 1)$, $E \hat{+} y$ is measurable and $m(E \hat{+} y) = m(E)$.

Proof. Define $E_1 = E \cap [0, 1 - y)$ and $E_2 = E \cap [1 - y, 1)$. Then $E_1 \cap E_2 = \emptyset$, $E = E_1 \cup E_2$, and $E_1, E_2 \in \mathcal{M}$. So $m(E) = m(E_1) + m(E_2)$ by countable additivity (Proposition 2.13). Now $E_1 \hat{+} y = E_1 + y$ and so $E_1 \hat{+} y \in \mathcal{M}$ and $m(E_1 \hat{+} y) = m(E)$ since $m$ is translation invariant (Proposition 2.2). Also, $E_2 \hat{+} y = (E_2 + y) - 1 = E_2 + (y - 1)$ and so $E_2 \hat{+} y \in \mathcal{M}$ and $m(E_2 \hat{+} y) = m(E_2)$. Next, $E \hat{+} y = (E_1 \hat{+} y) \cup (E_2 \hat{+} y)$ and $(E_1 \hat{+} y) \cap (E_2 \hat{+} y) = \emptyset$, so $E \hat{+} y \in \mathcal{M}$ and

$$m(E \hat{+} y) = m(E_1 \hat{+} y) + m(E_2 \hat{+} y) = m(E_1) + m(E_2) = m(E).$$
Lemma 2.6.A. Let $E \subset [0, 1)$ and $E \in \mathcal{M}$. Then for all $y \in [0, 1)$, $E \hat{+} y$ is measurable and $m(E \hat{+} y) = m(E)$.

**Proof.** Define $E_1 = E \cap [0, 1-y)$ and $E_2 = E \cap [1-y, 1)$. Then $E_1 \cap E_2 = \emptyset$, $E = E_1 \cup E_2$, and $E_1, E_2 \in \mathcal{M}$. So $m(E) = m(E_1) + m(E_2)$ by countable additivity (Proposition 2.13). Now $E_1 \hat{+} y = E_1 + y$ and so $E_1 \hat{+} y \in \mathcal{M}$ and $m(E_1 \hat{+} y) = m(E)$ since $m$ is translation invariant (Proposition 2.2). Also, $E_2 \hat{+} y = (E_2 + y) - 1 = E_2 + (y - 1)$ and so $E_2 \hat{+} y \in \mathcal{M}$ and $m(E_2 \hat{+} y) = m(E_2)$. Next, $E \hat{+} y = (E_1 \hat{+} y) \cup (E_2 \hat{+} y)$ and $(E_1 \hat{+} y) \cap (E_2 \hat{+} y) = \emptyset$, so $E \hat{+} y \in \mathcal{M}$ and

$$m(E \hat{+} y) = m(E_1 \hat{+} y) + m(E_2 \hat{+} y) = m(E_1) + m(E_2) = m(E).$$
Theorem 2.6.B. Set $P$ is not measurable.

Proof. First, we establish some set theoretic results. Let $\{r_i\}_{i=0}^\infty$ be an enumeration of $\mathbb{Q} \cap [0, 1)$ where $r_0 = 0$. Define $P_i = P + r_i$. Then $P_0 = P$. 
Theorem 2.6.B. Set $P$ is not measurable.

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If $x \in P_i \cap P_j$, then $x = p_i + r_i = p_j + r_j$ where $p_i, p_j \in P$. But then $p_i + (-p_j) = r_j + (-r_i) \in \mathbb{Q}$ and so $p_i \sim p_j$. Therefore $P_i \cap P_j = \emptyset$ if $i \neq j$ and so the $P_i$'s are disjoint and $\bigcup_{i=1}^{\infty} P_i \subset [0, 1)$. 
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If $x \in P_i \cap P_j$, then $x = p_i \hat{+} r_i = p_j \hat{+} r_j$ where $p_i, p_j \in P$. But then $p_i \hat{+} (-p_j) = r_j \hat{+} (-r_i) \in \mathbb{Q}$ and so $p_i \sim p_j$. Therefore $P_i \cap P_j = \emptyset$ if $i \neq j$ and so the $P_i$’s are disjoint and $\bigcup_{i=1}^{\infty} P_i \subset [0, 1)$.

Let $x \in [0, 1)$. Then $x$ is in some equivalence class $E_x$. Let $p_x \in P$ be the representative of class $E_x$ (i.e., $f(E_x) = p_x$ for choice function $f$).
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If $x \in P_i \cap P_j$, then $x = p_i \hat{+} r_i = p_j \hat{+} r_j$ where $p_i, p_j \in P$. But then $p_i \hat{+} (-p_j) = r_j \hat{+} (-r_i) \in \mathbb{Q}$ and so $p_i \sim p_j$. Therefore $P_i \cap P_j = \emptyset$ if $i \neq j$ and so the $P_i$’s are disjoint and $\bigcup_{i=1}^\infty P_i \subset [0,1)$.

Let $x \in [0,1)$. Then $x$ is in some equivalence class $E_x$. Let $p_x \in P$ be the representative of class $E_x$ (i.e., $f(E_x) = p_x$ for choice function $f$). Then $p_x \hat{+} q = x$ for some $q \in \mathbb{Q} \cap [0,1)$ and so $x \in \bigcup_{i=1}^\infty (P_i \hat{+} q_i) = \bigcup_{i=1}^\infty P_i$.

Hence, since $x$ is an arbitrary element of $[0,1)$ then $[0,1) \subset \bigcup_{i=1}^\infty P_i$. Therefore, $\bigcup_{i=1}^\infty P_i = [0,1)$. 


Theorem 2.6.B. Set $P$ is not measurable.

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If $x \in P_i \cap P_j$, then $x = p_i \hat{+} r_i = p_j \hat{+} r_j$ where $p_i, p_j \in P$. But then $p_i \hat{+} (-p_j) = r_j \hat{+} (-r_i) \in \mathbb{Q}$ and so $p_i \sim p_j$. Therefore $P_i \cap P_j = \emptyset$ if $i \neq j$ and so the $P_i$’s are disjoint and $\bigcup_{i=1}^{\infty} P_i \subset [0, 1)$.

Let $x \in [0, 1)$. Then $x$ is in some equivalence class $E_x$. Let $p_x \in P$ be the representative of class $E_x$ (i.e., $f(E_x) = p_x$ for choice function $f$). Then $p_x \hat{+} q = x$ for some $q \in \mathbb{Q} \cap [0, 1)$ and so $x \in \bigcup_{i=1}^{\infty} (P \hat{+} q_i) = \bigcup_{i=1}^{\infty} P_i$. Hence, since $x$ is an arbitrary element of $[0, 1)$ then $[0, 1) \subset \bigcup_{i=1}^{\infty} P_i$. Therefore, $\bigcup_{i=1}^{\infty} P_i = [0, 1)$. 
Theorem 2.6.B (continued)

**Theorem 2.6.B.** Set $P$ is not measurable.

**Proof (continued).** ASSUME $P$ is measurable. Then by Lemma 2.6.A, each $P_i$ is measurable and $m(P_i) = m(P)$. 

Hence $1 = m([0,1))$ by Propositions 2.1 and 2.8, $= m(\bigcup \bigcup_{i=1}^{\infty} P_i)$ since $[0,1) = \bigcup \bigcup_{i=1}^{\infty} P_i = \sum_{i=1}^{\infty} m(P_i)$ by countable additivity (Proposition 2.13), $= \sum_{i=1}^{\infty} m(P)$ since $m'(P) = m'(P_i)$ for all $i \in \mathbb{N} \cup \{0\}$. 

Therefore the assumption that $P$ is measurable is false and so $P$ is not measurable.
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**Proof (continued).** ASSUME $P$ is measurable. Then by Lemma 2.6.A, each $P_i$ is measurable and $m(P_i) = m(P)$. Hence

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1 = m([0, 1)) \text{ by Propositions 2.1 and 2.8}
\]
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= m(\bigcup_{i=1}^{\infty} P_i) \text{ since } [0, 1) = \bigcup_{i=1}^{\infty} P_i
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= \sum_{i=1}^{\infty} m(P_i) \text{ by countable additivity (Proposition 2.13)}
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\]
\[
= \begin{cases} 
0 & \text{if } m(P) = 0 \\
\infty & \text{if } m(P) > 0,
\end{cases}
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a CONTRADICTION.
Theorem 2.6.B. Set $P$ is not measurable.

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a CONTRADICTION. Therefore the assumption that \( P \) is measurable is false and so \( P \) is not measurable.
Theorem 2.18

There are disjoint sets of real numbers $A$ and $B$ for which

$$m^*(A \cup B) < m^*(A) + m^*(B).$$

Proof. Assume $m^*(A \cup B) = m^*(A) + m^*(B)$ for every disjoint pair of sets $A$ and $B$. 


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Proof. Assume $m^*(A \cup B) = m^*(A) + m^*(B)$ for every disjoint pair of sets $A$ and $B$. Then for any $A, E \subset \mathbb{R}$ we have

$$m^*(A) = m^*((A \cap E) \cup (A \cap E^c)) = m^*(A \cap E) + m^*(A \cap E^c)$$

and so every $E \subset \mathbb{R}$ is measurable, a CONTRADICTION to Corollary 2.6.C.
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There are disjoint sets of real numbers $A$ and $B$ for which

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Proof. ASSUME $m^*(A \cup B) = m^*(A) + m^*(B)$ for every disjoint pair of sets $A$ and $B$. Then for any $A, E \subset \mathbb{R}$ we have

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and so every $E \subset \mathbb{R}$ is measurable, a CONTRADICTION to Corollary 2.6.C. So for some disjoint $A, B \subset \mathbb{R}$ we have

$m^*(A \cup B) \neq m^*(A) + m^*(B)$. By subadditivity (Proposition 2.3) $m^*(A \cup B) \leq m^*(A) + m^*(B)$, so it must be that for some disjoint $A, B \subset \mathbb{R}$ we have $m^*(A \cup B) < m^*(A) + m^*(B)$.
Theorem 2.18

There are disjoint sets of real numbers $A$ and $B$ for which

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$m^*(A \cup B) \neq m^*(A) + m^*(B)$. By subadditivity (Proposition 2.3) $m^*(A \cup B) \leq m^*(A) + m^*(B)$, so it must be that for some disjoint $A, B \subset \mathbb{R}$ we have $m^*(A \cup B) < m^*(A) + m^*(B)$. □