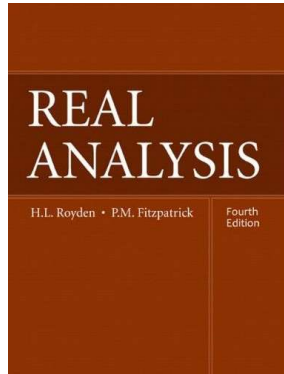


# Real Analysis

## Chapter 2. Lebesgue Measure

### 2.7. The Cantor Set and the Cantor-Lebesgue Function—Proofs of Theorems



## Proposition 2.19

**Proposition 2.19.** The Cantor set  $\mathbf{C}$  is a closed, uncountable set of measure zero.

**Proof.** Since  $\mathbf{C} = \bigcap_{k=1}^{\infty} C_k$  where each  $C_k$  is closed, then  $\mathbf{C}$  is closed (and so measurable).

Each  $C_k$  is the disjoint union of  $2^k$  intervals each of length  $1/3^k$ , so by countable additivity (Proposition 2.13)  $m(C_k) = (2/3)^k$ . By monotonicity of measure (Lemma 2.2.A),  $m(\mathbf{C}) \leq m(C_k) = (2/3)^k$  for all  $k \in \mathbb{N}$ , therefore  $m(\mathbf{C}) = 0$ .

ASSUME  $\mathbf{C}$  is countable. Let  $\{c_k\}_{k=1}^{\infty}$  be an enumeration of  $\mathbf{C}$ . Now  $C_1$  consists of two disjoint closed intervals, so one of them fails to contain point  $c_1$ ; denote it  $F_1$ . In  $C_2$ , there are two disjoint closed intervals which are subsets of  $F_1$ . One of these fails to contain point  $c_2$ ; denote it  $F_2$ . Similarly, recursively define sequence of sets  $\{F_k\}_{k=1}^{\infty}$ .

## Proposition 2.19

**Proposition 2.19.** The Cantor set  $\mathbf{C}$  is a closed, uncountable set of measure zero.

**Proof.** These sets satisfy: (i)  $F_k$  is closed and  $F_{k+1} \subset F_k$ , (ii)  $F_k \subset C_k$ , and (iii)  $c_k \notin F_k$ . From property (i) and the Nested Set Theorem (see page 19 of the book) we have that  $\bigcap_{k=1}^{\infty} F_k$  is nonempty, so let  $x \in \bigcap_{k=1}^{\infty} F_k$ . By property (ii),  $\bigcap_{k=1}^{\infty} F_k \subset \bigcap_{k=1}^{\infty} C_k = \mathbf{C}$ , so  $x \in \mathbf{C}$ . But  $\{c_k\}_{k=1}^{\infty} = \mathbf{C}$  so  $x = c_n$  for some  $n \in \mathbb{N}$ . Thus  $c_n = x = \bigcap_{k=1}^{\infty} F_k \subset F_n$  and so  $c_n \in F_n$ , a CONTRADICTION. So the assumption that  $\mathbf{C}$  is countable is false and therefore  $\mathbf{C}$  is uncountable.  $\square$

## Proposition 2.20

**Proposition 2.20.** The Cantor-Lebesgue function  $\varphi$  is an increasing continuous function that maps  $[0, 1]$  into  $[0, 1]$ . Its derivative exists on the open set  $\mathcal{O} = [0, 1] \setminus \mathbf{C}$  and  $\varphi'(x) = 0$  for  $x \in \mathcal{O}$ .

**Proof.** Since  $\varphi$  is increasing on  $\mathcal{O}$ , then for any  $u, v \in \mathcal{O}$  with  $u \leq v$  we have  $\varphi(u) \leq \varphi(v)$ . Now for  $u \leq v$  with  $u \in \mathcal{O}$  and  $v \in \mathbf{C} \setminus \{0\}$  we have  $\varphi(u) \leq \varphi(v) = \sup\{\varphi(t) \mid t \in \mathcal{O} \cap [0, v]\}$  since  $u \in \mathcal{O} \cap [0, v]$ . For  $u \leq v$  with  $u \in \mathbf{C} \setminus \{0\}$  and  $v \in \mathcal{O}$  we have  $\varphi(u) = \sup\{\varphi(t) \mid t \in \mathcal{O} \cap [0, u]\} \leq \varphi(v)$  since  $\varphi(t) \leq \varphi(v)$  for all  $t \in \mathcal{O} \cap [0, u]$ , because for such  $t$  we have  $t < u \leq v$  and  $\varphi$  is increasing on  $\mathcal{O}$ . For  $u < v$  with  $u, v \in \mathbf{C} \setminus \{0\}$  we have that there is some  $w \in \mathcal{O}$  with  $u < w < v$  and so  $\varphi(u) \leq \varphi(w) \leq \varphi(v)$  by the above arguments. Therefore,  $\varphi$  is an increasing function on  $[0, 1]$ .  
Next, continuity. . .

## Proposition 2.20 (continued 1)

**Proof (continued).**  $\varphi$  is continuous at each point of  $\mathcal{O}$  since  $\varphi$  is constant on each open interval component of  $\mathcal{O}$ . Now consider  $x_0 \in \mathbf{C}$  with  $x_0 \neq 0, 1$ . For  $k \in \mathbb{N}$  sufficiently large, we have that  $x_0$  lies between two consecutive open intervals in  $[0, 1] \setminus \mathbf{C}_k$ . Let  $a_k$  lie in the lower of these two components and  $b_k$  lie in the upper of these two components. Function  $\varphi$  is defined to increase by  $1/2^k$  across consecutive intervals in  $[0, 1] \setminus \mathbf{C}_k$ , therefore  $a_k < x_0 < b_k$  and  $\varphi(b_k) - \varphi(a_k) = 1/2^k$ . Now  $k$  can be arbitrarily large and such open interval components exist, so for given  $\varepsilon > 0$  if  $k \in \mathbb{N}$  is chosen such that  $\varepsilon < 1/2^k$  then for  $\delta > 0$  such that  $\delta < \min\{x_0 - a_k, b_k - x_0 \mid a_k \text{ and } b_k \text{ are as described above as elements of } [0, 1] \setminus \mathbf{C}_k\}$  then we have  $|x_0 - x| < \delta$  implies  $|\varphi(x_0) - \varphi(x)| \leq \varphi(b_k) - \varphi(a_k) = 1/2^k < \varepsilon$  (we are using the fact that  $\varphi$  is increasing here) and so  $\varphi$  is continuous at  $x_0 \in \mathbf{C} \setminus \{0, 1\}$ . Next,  $\varphi$  takes on the value  $1/2^k$  for  $x \in \mathbf{C}$  and  $x$  “near” 0 and so  $\varphi$  is continuous at  $x_0 = 0$  where  $\varphi(0) = 0$ ; if  $x_0 = 1$  then we know that  $\varphi$  takes on the values  $1 - 1/2^k$  for  $x \in \mathbf{C}$  and  $x$  “near” 1 and so  $\varphi$  is continuous at  $x_0 = 1$  where  $\varphi(1) = 1$ .

## Proposition 2.20 (continued 2)

**Proposition 2.20.** The Cantor-Lebesgue function  $\varphi$  is an increasing continuous function that maps  $[0, 1]$  into  $[0, 1]$ . Its derivative exists on the open set  $\mathcal{O} = [0, 1] \setminus \mathbf{C}$  and  $\varphi'(x) = 0$  for  $x \in \mathcal{O}$ .

**Proof (continued).** Since  $\varphi$  is constant on each of the open intervals in  $\mathcal{O}$ , then  $\varphi'(x) = 0$  for all  $x \in \mathcal{O}$ . Since  $\mathbf{C}$  has measure zero by Proposition 2.19 and so by the Excision Property (Lemma 2.4.A)  $m(\mathcal{O}) = m([0, 1] \setminus \mathbf{C}) = m([0, 1]) - m(\mathbf{C}) = 1$ .

Finally, since  $\varphi(0) = 0$ ,  $\varphi(1) = 1$ ,  $\varphi$  is increasing, and  $\varphi$  is continuous, then by the intermediate Value Theorem,  $\varphi$  maps  $[0, 1]$  onto  $[0, 1]$ .  $\square$

## Proposition 2.21

**Proposition 2.21.** Let  $\varphi$  be the Cantor-Lebesgue function and define the function  $\psi$  on  $[0, 1]$  by  $\psi(x) = \varphi(x) + x$ . Then  $\psi$  is a strictly increasing continuous function that maps  $[0, 1]$  onto  $[0, 2]$ ,

- (i) maps the Cantor set  $\mathbf{C}$  onto a measurable set of positive measure and
- (ii) maps a measurable set, a subset of the Cantor set, onto a nonmeasurable set.

**Proof.** Since  $\psi$  is the sum of two continuous increasing functions, one of which is strictly increasing, then  $\psi$  is continuous and strictly increasing. Since  $\psi(0) = 0$  and  $\psi(1) = 2$  then  $\psi([0, 1]) = [0, 2]$ . Now  $[0, 1] = \mathbf{C} \cup \mathcal{O}$  and since  $\psi$  is one to one then  $[0, 2] = \psi(\mathbf{C}) \cup \psi(\mathcal{O})$ . Now a strictly increasing continuous function defined on an interval has a continuous inverse (see Theorem 4-16 of my Analysis 1 [MATH 4217/5217] notes on [4.2. Monotone and Inverse Functions](#)). Therefore  $\psi(\mathbf{C})$  is closed and  $\psi(\mathcal{O})$  is open (inverse images of open/closed sets under a continuous function is open/closed; see Proposition 1.22) and so both are measurable.

## Proposition 2.21 (continued 1)

**Proof (continued).** Let  $\mathcal{O} = \cup_{k=1}^{\infty} I_k$  where the  $I_k$  are the connected components of  $\mathcal{O}$ . Then  $\varphi$  is constant on each  $I_k$  and so  $\psi$  maps  $I_k$  onto a translated copy of itself (translated by the constant given by  $\varphi$  on  $I_k$ ) of the same length (the “+x” part of  $\psi$  is the identity function). Since  $\psi$  is one to one, the collection  $\{\psi(I_k)\}_{k=1}^{\infty}$  is disjoint. By countable additivity (Proposition 2.13),

$$\begin{aligned} m(\psi(\mathcal{O})) &= m(\psi(\cup_{k=1}^{\infty} I_k)) = m(\cup_{k=1}^{\infty} \psi(I_k)) = \sum_{k=1}^{\infty} m(\psi(I_k)) \\ &= \sum_{k=1}^{\infty} \ell(\psi(I_k)) = \sum_{k=1}^{\infty} \ell(I_k) = \sum_{k=1}^{\infty} m(I_k) = m(\cup_{k=1}^{\infty} I_k) = m(\mathcal{O}). \end{aligned}$$

But  $m(\mathbf{C}) = 0$  and  $m(\mathcal{O}) = 1$ , so  $m(\psi(\mathcal{O})) = 1$ . Hence, since  $[0, 2] = \psi(\mathcal{O}) \cup \psi(\mathbf{C})$ , then  $m(\psi(\mathbf{C})) = 1$  and (i) follows.

## Proposition 2.21 (continued 2)

**Proposition 2.21.** Let  $\varphi$  be the Cantor-Lebesgue function and define the function  $\psi$  on  $[0, 1]$  by  $\psi(x) = \varphi(x) + x$ . Then  $\psi$  is a strictly increasing continuous function that maps  $[0, 1]$  onto  $[0, 2]$ ,

- (i) maps the Cantor set  $\mathbf{C}$  onto a measurable set of positive measure and
- (ii) maps a measurable set, a subset of the Cantor set, onto a nonmeasurable set.

**Proof (continued).** To verify (ii), notice that Vitali's Construction of a Nonmeasurable Set (Theorem 2.17) implies that  $\psi(\mathbf{C})$  contains a nonmeasurable subset  $W$ . The set  $\psi^{-1}(W)$  is measurable by Proposition 2.4 since  $\psi^{-1}(W) \subset \mathbf{C}$  and  $\mathbf{C}$  has measure 0 (so by monotonicity  $\psi^{-1}(W)$  has measure 0). So  $\psi^{-1}(W)$  is a measurable subset of  $\mathbf{C}$  which is mapped by  $\psi$  onto a nonmeasurable set.  $\square$

## Proposition 2.22

**Proposition 2.22.** There is a measurable set, a subset of the Cantor set, that is not a Borel set.

**Proof.** The strictly increasing continuous function  $\psi$  of Proposition 2.21 maps a measurable set  $A$  ( $A = \psi^{-1}(W)$  in the notation of the proof of Proposition 2.21) onto a nonmeasurable set  $B$  ( $B = W$  in the proof of Proposition 2.21). By Exercise 2.47, a continuous strictly increasing function that is defined on an interval maps Borel sets to Borel sets. So set  $A$  is not Borel, or else  $B = \psi(A)$  would be Borel and so measurable.  $\square$