Chapter 2. Lebesgue Measure
2.7. The Cantor Set and the Cantor-Lebesgue Function—Proofs of Theorems
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Proposition 2.19. The Cantor set $C$ is a closed, uncountable set of measure zero.

Proof. Since $C = \bigcap_{k=1}^{\infty} C_k$ where each $C_k$ is closed, then $C$ is closed (and so measurable).

Each $C_k$ is the disjoint union of $2^k$ intervals each of length $1/3^k$, so by countable additivity (Proposition 2.13) $m(C_k) = (2/3)^k$. By monotonicity of measure (Lemma 2.2.A), $m(C) \leq m(C_k) = (2/3)^k$ for all $k \in \mathbb{N}$, therefore $m(C) = 0$. 
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ASSUME $C$ is countable. Let $\{c_k\}_{k=1}^{\infty}$ be an enumeration of $C$. Now $C_1$ consists of two disjoint closed intervals, so one of them fails to contain point $c_1$; denote it $F_1$. 


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Proof. Since $C = \bigcap_{k=1}^{\infty} C_k$ where each $C_k$ is closed, then $C$ is closed (and so measurable).

Each $C_k$ is the disjoint union of $2^k$ intervals each of length $\frac{1}{3^k}$, so by countable additivity (Proposition 2.13) $m(C_k) = \left(\frac{2}{3}\right)^k$. By monotonicity of measure (Lemma 2.2.A), $m(C) \leq m(C_k) = \left(\frac{2}{3}\right)^k$ for all $k \in \mathbb{N}$, therefore $m(C) = 0$.

Assume $C$ is countable. Let $\{c_k\}_{k=1}^{\infty}$ be an enumeration of $C$. Now $C_1$ consists of two disjoint closed intervals, so one of them fails to contain point $c_1$; denote it $F_1$. In $C_2$, there are two disjoint closed intervals which are subsets of $F_1$. One of these fails to contain point $c_2$; denote it $F_2$. Similarly, recursively define sequence of sets $\{F_k\}_{k=1}^{\infty}$. 


Proposition 2.19

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Proposition 2.19. The Cantor set $\mathbb{C}$ is a closed, uncountable set of measure zero.

Proof. These sets satisfy: (i) $F_k$ is closed and $F_{k+1} \subset F_k$, (ii) $F_k \subset C_k$, and (iii) $c_k \not\in F_k$. 

From property (i) and the Nested Set Theorem (see page 19) we have that $\bigcap_{k=1}^{\infty} F_k$ is nonempty, so let $x \in \bigcap_{k=1}^{\infty} F_k$. By property (ii), $\bigcap_{k=1}^{\infty} F_k \subset \bigcap_{k=1}^{\infty} C_k = C$, so $x \in C$. But $\{c_k\}_{k=1}^{\infty} = C$ so $x = c_n$ for some $n \in \mathbb{N}$. Thus $c_n = x \in \bigcap_{k=1}^{\infty} F_k \subset F_n$ and so $c_n \in F_n$, a contradiction.

So the assumption that $C$ is countable is false and therefore $C$ is uncountable.
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Proposition 2.19. The Cantor set \( C \) is a closed, uncountable set of measure zero.

Proof. These sets satisfy: (i) \( F_k \) is closed and \( F_{k+1} \subset F_k \), (ii) \( F_k \subset C_k \), and (iii) \( c_k \notin F_k \). From property (i) and the Nested Set Theorem (see page 19) we have that \( \cap_{k=1}^{\infty} F_k \) is nonempty, so let \( x \in \cap_{k=1}^{\infty} F_k \). By property (ii), \( \cap_{k=1}^{\infty} F_k \subset \cap_{k=1}^{\infty} C_k = C \), so \( x \in C \). But \( \{c_k\}_{k=1}^{\infty} = C \) so \( x = c_n \) for some \( n \in \mathbb{N} \). Thus \( c_n = x = \cap_{k=1}^{\infty} F_k \subset F_n \) and so \( c_n \in F_n \), a CONTRADICTION.
Proposition 2.19. The Cantor set $\mathbf{C}$ is a closed, uncountable set of measure zero.

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**Proposition 2.19.** The Cantor set $\mathcal{C}$ is a closed, uncountable set of measure zero.

**Proof.** These sets satisfy: (i) $F_k$ is closed and $F_{k+1} \subset F_k$, (ii) $F_k \subset C_k$, and (iii) $c_k \not\in F_k$. From property (i) and the Nested Set Theorem (see page 19) we have that $\bigcap_{k=1}^{\infty} F_k$ is nonempty, so let $x \in \bigcap_{k=1}^{\infty} F_k$. By property (ii), $\bigcap_{k=1}^{\infty} F_k \subset \bigcap_{k=1}^{\infty} C_k = \mathcal{C}$, so $x \in \mathcal{C}$. But $\{c_k\}_{k=1}^{\infty} = \mathcal{C}$ so $x = c_n$ for some $n \in \mathbb{N}$. Thus $c_n = x = \bigcap_{k=1}^{\infty} F_k \subset F_n$ and so $c_n \in F_n$, a CONTRADICTION. So the assumption that $\mathcal{C}$ is countable is false and therefore $\mathcal{C}$ is uncountable. \qed
Proposition 2.20. The Cantor-Lebesgue function $\varphi$ is an increasing continuous function that maps $[0, 1]$ into $[0, 1]$. Its derivative exists on the open set $O = [0, 1] \setminus C$ and $\varphi'(x) = 0$ for $x \in O$.

Proof. Since $\varphi$ is increasing on $O$, then for any $u, v \in O$ with $u \leq v$ we have $\varphi(u) \leq \varphi(v)$. 
Proposition 2.20. The Cantor-Lebesgue function $\varphi$ is an increasing continuous function that maps $[0, 1]$ into $[0, 1]$. Its derivative exists on the open set $\mathcal{O} = [0, 1] \setminus \mathcal{C}$ and $\varphi'(x) = 0$ for $x \in \mathcal{O}$.

Proof. Since $\varphi$ is increasing on $\mathcal{O}$, then for any $u, v \in \mathcal{O}$ with $u \leq v$ we have $\varphi(u) \leq \varphi(v)$. Now for $u \leq v$ with $u \in \mathcal{O}$ and $v \in \mathcal{C} \setminus \{0\}$ we have $\varphi(u) \leq \varphi(v) = \sup\{\varphi(t) \mid t \in \mathcal{O} \cap [0, v)\}$ since $u \in \mathcal{O} \cap [0, v)$. 
Proposition 2.20. The Cantor-Lebesgue function $\varphi$ is an increasing continuous function that maps $[0, 1]$ into $[0, 1]$. Its derivative exists on the open set $\mathcal{O} = [0, 1] \setminus \mathbf{C}$ and $\varphi'(x) = 0$ for $x \in \mathcal{O}$.

Proof. Since $\varphi$ is increasing on $\mathcal{O}$, then for any $u, v \in \mathcal{O}$ with $u \leq v$ we have $\varphi(u) \leq \varphi(v)$. Now for $u \leq v$ with $u \in \mathcal{O}$ and $v \in \mathbf{C} \setminus \{0\}$ we have $\varphi(u) \leq \varphi(v) = \sup\{\varphi(t) \mid t \in \mathcal{O} \cap [0, v)\}$ since $u \in \mathcal{O} \cap [0, v)$. For $u \leq v$ with $u \in \mathbf{C} \setminus \{0\}$ and $v \in \mathcal{O}$ we have $\varphi(u) = \sup\{\varphi(t) \mid t \in \mathcal{O} \cap [0, u)\} \leq \varphi(v)$ since $\varphi(t) \leq \varphi(v)$ for all $t \in \mathcal{O} \cap [0, u)$ since for such $t$ we have $t < u \leq v$ and $\varphi$ is increasing on $\mathcal{O}$. 
Proposition 2.20. The Cantor-Lebesgue function \( \varphi \) is an increasing continuous function that maps \([0, 1]\) into \([0, 1]\). Its derivative exists on the open set \( \mathcal{O} = [0, 1] \setminus \mathcal{C} \) and \( \varphi'(x) = 0 \) for \( x \in \mathcal{O} \).

Proof. Since \( \varphi \) is increasing on \( \mathcal{O} \), then for any \( u, v \in \mathcal{O} \) with \( u \leq v \) we have \( \varphi(u) \leq \varphi(v) \). Now for \( u \leq v \) with \( u \in \mathcal{O} \) and \( v \in \mathcal{C} \setminus \{0\} \) we have

\[
\varphi(u) \leq \varphi(v) = \sup\{\varphi(t) \mid t \in \mathcal{O} \cap [0, v)\} \quad \text{since} \quad u \in \mathcal{O} \cap [0, v).
\]

For \( u \leq v \) with \( u \in \mathcal{C} \setminus \{0\} \) and \( v \in \mathcal{O} \) we have

\[
\varphi(u) = \sup\{\varphi(t) \mid t \in \mathcal{O} \cap [0, u)\} \leq \varphi(v) \quad \text{since} \quad \varphi(t) \leq \varphi(v) \quad \text{for all} \quad t \in \mathcal{O} \cap [0, u)
\]

since for such \( t \) we have \( t < u \leq v \) and \( \varphi \) is increasing on \( \mathcal{O} \). For \( u < v \) with \( u, v \in \mathcal{C} \setminus \{0\} \) we have that there is some \( w \in \mathcal{O} \) with \( u < w < v \) and so \( \varphi(u) \leq \varphi(w) \leq \varphi(v) \) by the above arguments. Therefore, \( \varphi \) is an increasing function on \([0, 1]\).
**Proposition 2.20.** The Cantor-Lebesgue function $\varphi$ is an increasing continuous function that maps $[0, 1]$ into $[0, 1]$. Its derivative exists on the open set $\mathcal{O} = [0, 1] \setminus C$ and $\varphi'(x) = 0$ for $x \in \mathcal{O}$.

**Proof.** Since $\varphi$ is increasing on $\mathcal{O}$, then for any $u, v \in \mathcal{O}$ with $u \leq v$ we have $\varphi(u) \leq \varphi(v)$. Now for $u \leq v$ with $u \in \mathcal{O}$ and $v \in C \setminus \{0\}$ we have $\varphi(u) \leq \varphi(v) = \sup\{\varphi(t) \mid t \in \mathcal{O} \cap [0, v]\}$ since $u \in \mathcal{O} \cap [0, v)$. For $u \leq v$ with $u \in C \setminus \{0\}$ and $v \in \mathcal{O}$ we have $\varphi(u) = \sup\{\varphi(t) \mid t \in \mathcal{O} \cap [0, u]\} \leq \varphi(v)$ since $\varphi(t) \leq \varphi(v)$ for all $t \in \mathcal{O} \cap [0, u)$ since for such $t$ we have $t < u \leq v$ and $\varphi$ is increasing on $\mathcal{O}$. For $u < v$ with $u, v \in C \setminus \{0\}$ we have that there is some $w \in \mathcal{O}$ with $u < w < v$ and so $\varphi(u) \leq \varphi(w) \leq \varphi(v)$ by the above arguments. Therefore, $\varphi$ is an increasing function on $[0, 1]$. 
Proposition 2.20 (continued 1)

Proof (continued). Next, for continuity. \( \phi \) is continuous at each point of \( \mathcal{O} \) since \( \phi \) is constant on each constituent open interval component of \( \mathcal{O} \). Now consider \( x_0 \in C \) with \( x_0 \neq 0, 1 \). For \( k \in \mathbb{N} \) sufficiently large, we have that \( x_0 \) lies between two consecutive open intervals in \([0, 1] \setminus C_k \). Let \( a_k \) lie in the lower of these two components and \( b_k \) lie in the upper of these two components.
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Proof (continued). Next, for continuity. \( \varphi \) is continuous at each point of \( O \) since \( \varphi \) is constant on each constituent open interval component of \( O \). Now consider \( x_0 \in C \) with \( x_0 \neq 0, 1 \). For \( k \in \mathbb{N} \) sufficiently large, we have that \( x_0 \) lies between two consecutive open intervals in \([-1, 1] \setminus C_k \). Let \( a_k \) lie in the lower of these two components and \( b_k \) lie in the upper of these two components. Function \( \varphi \) is defined to increase by \( 1/2^k \) across consecutive intervals in \([-1, 1] \setminus C_k \), therefore \( a_k < x_0 < b_k \) and \( \varphi(b_k) - \varphi(a_k) = 1/2^k \). Now \( k \) can be arbitrarily large and such open interval components exist, so for given \( \varepsilon > 0 \) it \( k \in \mathbb{N} \) is chosen such that \( \varepsilon < 1/2^k \) then for \( \delta > 0 \) such that \( \delta < \min\{x_0 - a_k, b_k = x_0 \mid a_k \text{ and } b_k \text{ are as described above as elements of } [-1, 1] \setminus C_k\} \) then we have \( |x_0 - x| < \delta \) implies \( |\varphi(x_0) - \varphi(x)| \leq |\varphi(b_k) - \varphi(a_k)| = 1/2^k < \varepsilon \) (we are using the fact that \( \varphi \) is increasing here) and so \( \varphi \) takes on the value \( 1/2^k \) for \( x \in C \) and \( x \) “near” 0 and so \( \varphi \) is continuous at \( x_0 = 0 \) where \( \varphi(0) = 0 \);
Proposition 2.20 (continued 1)

**Proof (continued).** Next, for continuity. \( \varphi \) is continuous at each point of \( \mathcal{O} \) since \( \varphi \) is constant on each constituent open interval component of \( \mathcal{O} \). Now consider \( x_0 \in \mathbb{C} \) with \( x_0 \neq 0, 1 \). For \( k \in \mathbb{N} \) sufficiently large, we have that \( x_0 \) lies between two consecutive open intervals in \( [0, 1] \setminus C_k \). Let \( a_k \) lie in the lower of these two components and \( b_k \) lie in the upper of these two components. Function \( \varphi \) is defined to increase by \( 1/2^k \) across consecutive intervals in \( [0, 1] \setminus C_k \), therefore \( a_k < x_0 < b_k \) and \( \varphi(b_k) - \varphi(a_k) = 1/2^k \).

Now \( k \) can be arbitrarily large and such open interval components exist, so for given \( \varepsilon > 0 \) it \( k \in \mathbb{N} \) is chosen such that \( \varepsilon < 1/2^k \) then for \( \delta > 0 \) such that \( \delta < \min\{x_0 - a_k, b_k = x_0 \mid a_k \text{ and } b_k \text{ are as described above as elements of } [0, 1] \setminus C_k\} \) then we have \( |x_0 - x| < \delta \) implies \( |\varphi(x_0) - \varphi(x)| \leq \varphi(b_k) - \varphi(a_k) = 1/2^k < \varepsilon \) (we are using the fact that \( \varphi \) is increasing here) and so \( \varphi \) takes on the value \( 1/2^k \) for \( x \in \mathbb{C} \) and \( x \) “near” 0 and so \( \varphi \) is continuous at \( x_0 = 0 \) where \( \varphi(0) = 0 \); if \( x_0 = 1 \) then we know that \( \varphi \) takes on the values \( 1 - 1/2^k \) for \( x \in \mathbb{C} \) and \( x \) “near” 1 and so \( f \) is continuous at \( x_0 = 1 \) where \( \varphi(1) = 1 \).
Proposition 2.20 (continued 1)

Proof (continued). Next, for continuity. \( \varphi \) is continuous at each point of \( \mathcal{O} \) since \( \varphi \) is constant on each constituent open interval component of \( \mathcal{O} \). Now consider \( x_0 \in \mathbb{C} \) with \( x_0 \neq 0, 1 \). For \( k \in \mathbb{N} \) sufficiently large, we have that \( x_0 \) lies between two consecutive open intervals in \([0, 1] \setminus C_k\). Let \( a_k \) lie in the lower of these two components and \( b_k \) lie in the upper of these two components. Function \( \varphi \) is defined to increase by \( 1/2^k \) across consecutive intervals in \([0, 1] \setminus C_k\), therefore \( a_k < x_0 < b_k \) and \( \varphi(b_k) - \varphi(a_k) = 1/2^k \).

Now \( k \) can be arbitrarily large and such open interval components exist, so for given \( \varepsilon > 0 \) it \( k \in \mathbb{N} \) is chosen such that \( \varepsilon < 1/2^k \) then for \( \delta > 0 \) such that \( \delta < \min \{ x_0 - a_k, b_k - x_0 \mid a_k \text{ and } b_k \text{ are as described} \} \) then we have \( |x_0 - x| < \delta \) implies \( |\varphi(x_0) - \varphi(x)| \leq \varphi(b_k) - \varphi(a_k) = 1/2^k < \varepsilon \) (we are using the fact that \( \varphi \) is increasing here) and so \( \varphi \) takes on the value \( 1/2^k \) for \( x \in \mathbb{C} \) and \( x \) “near” 0 and so \( \varphi \) is continuous at \( x_0 = 0 \) where \( \varphi(0) = 0 \); if \( x_0 = 1 \) then we know that \( \varphi \) takes on the values \( 1 - 1/2^k \) for \( x \in \mathbb{C} \) and \( x \) “near” 1 and so \( f \) is continuous at \( x_0 = 1 \) where \( \varphi(1) = 1 \).
Proposition 2.20. The Cantor-Lebesgue function $\varphi$ is an increasing continuous function that maps $[0, 1]$ into $[0, 1]$. Its derivative exists on the open set $\mathcal{O} = [0, 1] \setminus \mathbf{C}$ and $\varphi'(x) = 0$ for $x \in \mathcal{O}$.

Proof (continued). Since $\varphi$ is constant on each of the open intervals in $\mathcal{O}$, then $\varphi'(x) = 0$ for all $x \in \mathcal{O}$. Since $\mathbf{C}$ has measure zero by Proposition 2.19 and so by the Excision Property (Lemma 2.4.A)

$$m(\mathcal{O}) = m([0, 1] \setminus \mathbf{C}) = m([0, 1]) - m(\mathbf{C}) = 1.$$ 

Finally, since $\varphi(0) = 0$, $\varphi(1) = 1$, $\varphi$ is increasing, and $\varphi$ is continuous, then by the intermediate Value Theorem, $\varphi$ maps $[0, 1]$ onto $[0, 1]$. $\square$
Proposition 2.20. The Cantor-Lebesgue function $\varphi$ is an increasing continuous function that maps $[0, 1]$ into $[0, 1]$. Its derivative exists on the open set $\mathcal{O} = [0, 1] \setminus C$ and $\varphi'(x) = 0$ for $x \in \mathcal{O}$.

Proof (continued). Since $\varphi$ is constant on each of the open intervals in $\mathcal{O}$, then $\varphi'(x) = 0$ for all $x \in \mathcal{O}$. Since $C$ has measure zero by Proposition 2.19 and so by the Excision Property (Lemma 2.4.A)

$$m(\mathcal{O}) = m([0, 1] \setminus C) = m([0, 1]) - m(C) = 1.$$  

Finally, since $\varphi(0) = 0$, $\varphi(1) = 1$, $\varphi$ is increasing, and $\varphi$ is continuous, then by the intermediate Value Theorem, $\varphi$ maps $[0, 1]$ onto $[0, 1]$. \qed
Proposition 2.21. Let \( \varphi \) be the Cantor-Lebesgue function and define the function \( \psi \) on \([0, 1]\) by \( \psi(x) = \varphi(x) + x \). Then \( \psi \) is a strictly increasing continuous function that maps \([0, 1]\) onto \([0, 2]\),

(i) maps the Cantor set \( C \) onto a measurable set of positive measure and

(ii) maps a measurable set, a subset of the Cantor set, onto a nonmeasurable set.

**Proof.** Since \( \psi \) is the sum of two continuous increasing functions, one of which is strictly increasing, then \( \psi \) is continuous and strictly increasing. Since \( \psi(0) = 0 \) and \( \psi(1) = 2 \) then \( \psi([0, 1]) = [0, 2] \).
Proposition 2.21. Let \( \varphi \) be the Cantor-Lebesgue function and define the function \( \psi \) on \([0, 1]\) by \( \psi(x) = \varphi(x) + x \). Then \( \psi \) is a strictly increasing continuous function that maps \([0, 1]\) onto \([0, 2]\),

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Proof. Since \( \psi \) is the sum of two continuous increasing functions, one of which is strictly increasing, then \( \psi \) is continuous and strictly increasing. Since \( \psi(0) = 0 \) and \( \psi(1) = 2 \) then \( \psi([0, 1]) = [0, 2] \). Now \([0, 1] = C \cup \mathcal{O}\) and since \( \psi \) is one to one then \([0, 2] = \psi(C \cup \psi(\mathcal{O})\). Now a strictly increasing continuous function defined on an interval has a continuous inverse (see Theorem 4-16 of my Analysis 1 [MATH 4217/5217] notes for Section 4.2). Therefore \( \psi(C) \) is closed and \( \psi(\mathcal{O}) \) is open (inverse images of open/closed sets under a continuous function is open/closed; see Proposition 1.22) and so both are measurable.
Proposition 2.21

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(i) maps the Cantor set \( C \) onto a measurable set of positive measure and

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Proof. Since \( \psi \) is the sum of two continuous increasing functions, one of which is strictly increasing, then \( \psi \) is continuous and strictly increasing. Since \( \psi(0) = 0 \) and \( \psi(1) = 2 \) then \( \psi([0, 1]) = [0, 2] \). Now \( [0, 1] = C \cup O \) and since \( \psi \) is one to one then \( [0, 2] = \psi(C \cup \psi(O)) \). Now a strictly increasing continuous function defined on an interval has a continuous inverse (see Theorem 4-16 of my Analysis 1 [MATH 4217/5217] notes for Section 4.2). Therefore \( \psi(C) \) is closed and \( \psi(O) \) is open (inverse images of open/closed sets under a continuous function is open/closed; see Proposition 1.22) and so both are measurable.
Proposition 2.21 (continued 1)

**Proof (continued).** Let \( \mathcal{O} = \bigcup_{k=1}^{\infty} I_k \) where the \( I_k \) are the connected components of \( \mathcal{O} \). Then \( \varphi \) is constant on each \( I_k \) and so \( \varphi \) maps \( I_k \) onto a translated copy of itself (translated by the constant given by \( \varphi \) on \( I_k \)) of the same length (the “+x” part of \( \psi \) is the identity function).

Since \( \psi \) is one to one, the collection \( \{\psi(I_k)\}_{k=1}^{\infty} \) is disjoint. By countable additivity (Proposition 2.13),

\[
m(\psi(\mathcal{O})) = m(\psi(\bigcup_{k=1}^{\infty} I_k)) = m(\bigcup_{k=1}^{\infty} \psi(I_k)) = \sum_{k=1}^{\infty} m(\psi(I_k))
\]

\[
= \sum_{k=1}^{\infty} \ell(\psi(I_k)) = \sum_{k=1}^{\infty} + \sum_{k=1}^{\infty} \ell(I_k) = \sum_{k=1}^{\infty} m(I_k) = m(\bigcup_{k=1}^{\infty} I_k) = m(\mathcal{O}).
\]
Proposition 2.21 (continued 1)

**Proof (continued).** Let \( \mathcal{O} = \bigcup_{k=1}^{\infty} I_k \) where the \( I_k \) are the connected components of \( \mathcal{O} \). Then \( \varphi \) is constant on each \( I_k \) and so \( \varphi \) maps \( I_k \) onto a translated translated copy of itself (translated by the constant given by \( \varphi \) on \( I_k \)) of the same length (the “+\( \chi \)” part of \( \psi \) is the identity function). Since \( \psi \) is one to one, the collection \( \{\psi(I_k)\}_{k=1}^{\infty} \) is disjoint. By countable additivity (Proposition 2.13),

\[
m(\psi(\mathcal{O})) = m(\psi(\bigcup_{k=1}^{\infty} I_k)) = m(\bigcup_{k=1}^{\infty} \psi(I_k)) = \sum_{k=1}^{\infty} m(\psi(I_k))
\]

\[
= \sum_{k=1}^{\infty} \ell(\psi(I_k)) = \sum_{k=1}^{\infty} +\chi(I_k) = \sum_{k=1}^{\infty} m(I_k) = m(\bigcup_{k=1}^{\infty} I_k) = m(\mathcal{O}).
\]

But \( m(C) = 0 \) and so \( m(\mathcal{O}) = 1 \), so \( m(\psi(\mathcal{O})) = 1 \). Hence, since \( [0, 2] = \psi(\mathcal{O}) \cup \psi(C) \), then \( m(\psi(C)) = 1 \) and (i) follows.
**Proposition 2.21 (continued 1)**

**Proof (continued).** Let $\mathcal{O} = \bigcup_{k=1}^{\infty} I_k$ where the $I_k$ are the connected components of $\mathcal{O}$. Then $\varphi$ is constant on each $I_k$ and so $\varphi$ maps $I_k$ onto a translated translated copy of itself (translated by the constant given by $\varphi$ on $I_k$) of the same length (the “$+x$” part of $\psi$ is the identity function). Since $\psi$ is one to one, the collection $\{\psi(I_k)\}_{k=1}^{\infty}$ is disjoint. By countable additivity (Proposition 2.13),

$$m(\psi(\mathcal{O})) = m(\psi(\bigcup_{k=1}^{\infty} I_k)) = m(\bigcup_{k=1}^{\infty} \psi(I_k)) = \sum_{k=1}^{\infty} m(\psi(I_k))$$

$$= \sum_{k=1}^{\infty} \ell(\psi(I_k)) = \sum_{k=1}^{\infty} \ell(I_k) = \sum_{k=1}^{\infty} m(I_k) = m(\bigcup_{k=1}^{\infty} I_k) = m(\mathcal{O}).$$

But $m(C) = 0$ and so $m(\mathcal{O}) = 1$, so $m(\psi(\mathcal{O})) = 1$. Hence, since $[0, 2] = \psi(\mathcal{O}) \cup \psi(C)$, then $m(\psi(C)) = 1$ and (i) follows.
Proposition 2.21. Let $\varphi$ be the Cantor-Lebesgue function and define the function $\psi$ on $[0,1]$ by $\psi(x) = \varphi(x) + x$. Then $\psi$ is a strictly increasing continuous function that maps $[0,1]$ onto $[0,2]$,

(i) maps the Cantor set $\mathbb{C}$ onto a measurable set of positive measure and

(ii) maps a measurable set, a subset of the Cantor set, onto a nonmeasurable set.

Proof (continued). To verify (ii), notice that Vitali’s Construction of a Nonmeasurable Set (Theorem 2.17) implies that $\psi(\mathbb{C})$ contains a nonmeasurable subset $W$. The set $\psi^{-1}(W)$ is measurable by Proposition 2.4 since $\psi^{-1}(W) \subset \mathbb{C}$ and $\mathbb{C}$ has measure 0 (so by monotonicity $\psi^{-1}(W)$ has measure 0). So $\psi^{-1}(W)$ is a measurable subset of $\mathbb{C}$ which is mapped by $\psi$ onto a nonmeasurable set.
Proposition 2.21. Let $\varphi$ be the Cantor-Lebesgue function and define the function $\psi$ on $[0,1]$ by $\psi(x) = \varphi(x) + x$. Then $\psi$ is a strictly increasing continuous function that maps $[0,1]$ onto $[0,2]$, 

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Proof (continued). To verify (ii), notice that Vitali’s Construction of a Nonmeasurable Set (Theorem 2.17) implies that $\psi(\mathcal{C})$ contains a nonmeasurable subset $W$. The set $\psi^{-1}(W)$ is measurable by Proposition 2.4 since $\psi^{-1}(W) \subset \mathcal{C}$ and $\mathcal{C}$ has measure 0 (so by monotonicity $\psi^{-1}(W)$ has measure 0). So $\psi^{-1}(W)$ is a measurable subset of $\mathcal{C}$ which is mapped by $\psi$ onto a nonmeasurable set.
Proposition 2.22. There is a measurable set, a subset of the Cantor set, that is not a Borel set.

Proof. The strictly increasing continuous function $\psi$ of Proposition 2.21 maps a measurable set $A$ ($A = \psi^{-1}(W)$ in the notation of the proof of Proposition 2.21) onto a measurable set $B$ ($B = W$ in the proof of Proposition 2.21).
Proposition 2.22. There is a measurable set, a subset of the Cantor set, that is not a Borel set.

Proof. The strictly increasing continuous function $\psi$ of Proposition 2.21 maps a measurable set $A$ ($A = \psi^{-1}(W)$ in the notation of the proof of Proposition 2.21) onto a measurable set $B$ ($B = W$ in the proof of Proposition 2.21). By Exercise 2.47, a continuous strictly increasing function that is defined on an interval maps Borel sets to Borel sets. So set $A$ is not Borel, or else $B = \psi(A)$ would be Borel and so measurable. \hfill $\blacksquare$
Proposition 2.22. There is a measurable set, a subset of the Cantor set, that is not a Borel set.

Proof. The strictly increasing continuous function $\psi$ of Proposition 2.21 maps a measurable set $A$ ($A = \psi^{-1}(W)$ in the notation of the proof of Proposition 2.21) onto a measurable set $B$ ($B = W$ in the proof of Proposition 2.21). By Exercise 2.47, a continuous strictly increasing function that is defined on an interval maps Borel sets to Borel sets. So set $A$ is not Borel, or else $B = \psi(A)$ would be Borel and so measurable. \qed