Real Analysis

Chapter 2. Lebesgue Measure

2.7. The Cantor Set and the Cantor-Lebesgue Function—Proofs of Theorems

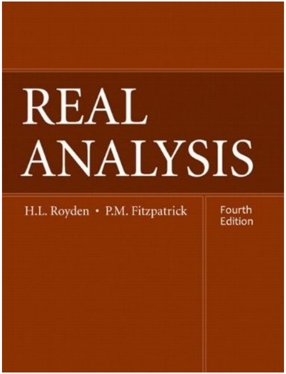

Table of contents

Proposition 2.19

Proposition 2.19. The Cantor set **C** is a closed, uncountable set of measure zero.

Proof. Since $\mathbf{C} = \cap_{k=1}^{\infty} C_k$ where each C_k is closed, then **C** is closed (and so measurable).

Proposition 2.19

Proposition 2.19. The Cantor set $\mathbf{C}$ is a closed, uncountable set of measure zero.

Proof. Since $\mathbf{C} = \cap_{k=1}^{\infty} C_k$ where each C_k is closed, then $\mathbf{C}$ is closed (and so measurable).

Each C_k is the disjoint union of 2^k intervals each of length $1/3^k$, so by countable additivity (Proposition 2.13) $m(C_k) = (2/3)^k$. By monotonicity of measure (Lemma 2.2.A), $m(\mathbf{C}) \leq m(C_k) = (2/3)^k$ for all $k \in \mathbb{N}$, therefore $m(\mathbf{C}) = 0$.

Proposition 2.19

Proposition 2.19. The Cantor set $\mathbf{C}$ is a closed, uncountable set of measure zero.

Proof. Since $\mathbf{C} = \cap_{k=1}^{\infty} C_k$ where each C_k is closed, then $\mathbf{C}$ is closed (and so measurable).

Each C_k is the disjoint union of 2^k intervals each of length $1/3^k$, so by countable additivity (Proposition 2.13) $m(C_k) = (2/3)^k$. By monotonicity of measure (Lemma 2.2.A), $m(\mathbf{C}) \leq m(C_k) = (2/3)^k$ for all $k \in \mathbb{N}$, therefore $m(\mathbf{C}) = 0$.

ASSUME $\mathbf{C}$ is countable. Let $\{c_k\}_{k=1}^{\infty}$ be an enumeration of $\mathbf{C}$. Now C_1 consists of two disjoint closed intervals, so one of them fails to contain point c_1; denote it F_1.

Proposition 2.19

Proposition 2.19. The Cantor set $\mathbf{C}$ is a closed, uncountable set of measure zero.

Proof. Since $\mathbf{C} = \cap_{k=1}^{\infty} C_k$ where each C_k is closed, then $\mathbf{C}$ is closed (and so measurable).

Each C_k is the disjoint union of 2^k intervals each of length $1/3^k$, so by countable additivity (Proposition 2.13) $m(C_k) = (2/3)^k$. By monotonicity of measure (Lemma 2.2.A), $m(\mathbf{C}) \le m(C_k) = (2/3)^k$ for all $k \in \mathbb{N}$, therefore $m(\mathbf{C}) = 0$.

ASSUME $\mathbf{C}$ is countable. Let $\{c_k\}_{k=1}^{\infty}$ be an enumeration of $\mathbf{C}$. Now C_1 consists of two disjoint closed intervals, so one of them fails to contain point c_1; denote it F_1. In C_2, there are two disjoint closed intervals which are subsets of F_1. One of these fails to contain point c_2; denote it F_2. Similarly, recursively define sequence of sets $\{F_k\}_{k=1}^{\infty}$.

Proposition 2.19

Proposition 2.19. The Cantor set $\mathbf{C}$ is a closed, uncountable set of measure zero.

Proof. Since $\mathbf{C} = \cap_{k=1}^{\infty} C_k$ where each C_k is closed, then $\mathbf{C}$ is closed (and so measurable).

Each C_k is the disjoint union of 2^k intervals each of length $1/3^k$, so by countable additivity (Proposition 2.13) $m(C_k) = (2/3)^k$. By monotonicity of measure (Lemma 2.2.A), $m(\mathbf{C}) \leq m(C_k) = (2/3)^k$ for all $k \in \mathbb{N}$, therefore $m(\mathbf{C}) = 0$.

ASSUME $\mathbf{C}$ is countable. Let $\{c_k\}_{k=1}^{\infty}$ be an enumeration of $\mathbf{C}$. Now C_1 consists of two disjoint closed intervals, so one of them fails to contain point c_1; denote it F_1. In C_2, there are two disjoint closed intervals which are subsets of F_1. One of these fails to contain point c_2; denote it F_2. Similarly, recursively define sequence of sets $\{F_k\}_{k=1}^{\infty}$.

Proposition 2.19

Proposition 2.19. The Cantor set **C** is a closed, uncountable set of measure zero.

Proof. These sets satisfy: (i) F_k is closed and $F_{k+1} \subset F_k$, (ii) $F_k \subset C_k$, and (iii) $c_k \notin F_k$.

Proposition 2.19

Proposition 2.19. The Cantor set $\mathbf{C}$ is a closed, uncountable set of measure zero.

Proof. These sets satisfy: (i) F_k is closed and $F_{k+1} \subset F_k$, (ii) $F_k \subset C_k$, and (iii) $c_k \notin F_k$. From property (i) and the Nested Set Theorem (see page 19 of the book) we have that $\cap_{k=1}^{\infty} F_k$ is nonempty, so let $x \in \cap_{k=1}^{\infty} F_k$. By property (ii), $\cap_{k=1}^{\infty} F_k \subset \cap_{k=1}^{\infty} C_k = \mathbf{C}$, so $x \in \mathbf{C}$.

Proposition 2.19

Proposition 2.19. The Cantor set $\mathbf{C}$ is a closed, uncountable set of measure zero.

Proof. These sets satisfy: (i) F_k is closed and $F_{k+1} \subset F_k$, (ii) $F_k \subset C_k$, and (iii) $c_k \notin F_k$. From property (i) and the Nested Set Theorem (see page 19 of the book) we have that $\cap_{k=1}^{\infty} F_k$ is nonempty, so let $x \in \cap_{k=1}^{\infty} F_k$. By property (ii), $\cap_{k=1}^{\infty} F_k \subset \cap_{k=1}^{\infty} C_k = \mathbf{C}$, so $x \in \mathbf{C}$. But $\{c_k\}_{k=1}^{\infty} = \mathbf{C}$ so $x = c_n$ for some $n \in \mathbb{N}$. Thus $c_n = x \in \cap_{k=1}^{\infty} F_k \subset F_n$ and so $c_n \in F_n$, a CONTRADICTION.

Proposition 2.19

Proposition 2.19. The Cantor set $\mathbf{C}$ is a closed, uncountable set of measure zero.

Proof. These sets satisfy: (i) F_k is closed and $F_{k+1} \subset F_k$, (ii) $F_k \subset C_k$, and (iii) $c_k \notin F_k$. From property (i) and the Nested Set Theorem (see page 19 of the book) we have that $\cap_{k=1}^{\infty} F_k$ is nonempty, so let $x \in \cap_{k=1}^{\infty} F_k$. By property (ii), $\cap_{k=1}^{\infty} F_k \subset \cap_{k=1}^{\infty} C_k = \mathbf{C}$, so $x \in \mathbf{C}$. But $\{c_k\}_{k=1}^{\infty} = \mathbf{C}$ so $x = c_n$ for some $n \in \mathbb{N}$. Thus $c_n = x \in \cap_{k=1}^{\infty} F_k \subset F_n$ and so $c_n \in F_n$, a CONTRADICTION. So the assumption that $\mathbf{C}$ is countable is false and therefore $\mathbf{C}$ is uncountable. □

Proposition 2.19

Proposition 2.19. The Cantor set **C** is a closed, uncountable set of measure zero.

Proof. These sets satisfy: (i) F_k is closed and $F_{k+1} \subset F_k$, (ii) $F_k \subset C_k$, and (iii) $c_k \notin F_k$. From property (i) and the Nested Set Theorem (see page 19 of the book) we have that $\cap_{k=1}^{\infty} F_k$ is nonempty, so let $x \in \cap_{k=1}^{\infty} F_k$. By property (ii), $\cap_{k=1}^{\infty} F_k \subset \cap_{k=1}^{\infty} C_k = \mathbf{C}$, so $x \in \mathbf{C}$. But $\{c_k\}_{k=1}^{\infty} = \mathbf{C}$ so $x = c_n$ for some $n \in \mathbb{N}$. Thus $c_n = x \in \cap_{k=1}^{\infty} F_k \subset F_n$ and so $c_n \in F_n$, a CONTRADICTION. So the assumption that **C** is countable is false and therefore **C** is uncountable. □

Proposition 2.20

Proposition 2.20. The Cantor-Lebesgue function φ is an increasing continuous function that maps $[0,1]$ onto $[0,1]$. Its derivative exists on the open set $\mathcal{O} = [0,1] \setminus \mathbf{C}$ and $\varphi'(x) = 0$ for $x \in \mathcal{O}$.

Proof. Since φ is increasing on $\mathcal{O}$, then for any $u, v \in \mathcal{O}$ with $u \leq v$ we have $\varphi(u) \leq \varphi(v)$.

Proposition 2.20

Proposition 2.20. The Cantor-Lebesgue function φ is an increasing continuous function that maps $[0,1]$ onto $[0,1]$. Its derivative exists on the open set $\mathcal{O} = [0,1] \setminus \mathbf{C}$ and $\varphi'(x) = 0$ for $x \in \mathcal{O}$.

Proof. Since φ is increasing on $\mathcal{O}$, then for any $u, v \in \mathcal{O}$ with $u \leq v$ we have $\varphi(u) \leq \varphi(v)$. Now for $u \leq v$ with $u \in \mathcal{O}$ and $v \in \mathbf{C} \setminus \{0\}$ we have $\varphi(u) \leq \varphi(v) = \sup\{\varphi(t) \mid t \in \mathcal{O} \cap [0,v)\}$ since $u \in \mathcal{O} \cap [0,v)$.

Proposition 2.20

Proposition 2.20. The Cantor-Lebesgue function φ is an increasing continuous function that maps $[0,1]$ onto $[0,1]$. Its derivative exists on the open set $\mathcal{O} = [0,1] \setminus \mathbf{C}$ and $\varphi'(x) = 0$ for $x \in \mathcal{O}$.

Proof. Since φ is increasing on $\mathcal{O}$, then for any $u, v \in \mathcal{O}$ with $u \leq v$ we have $\varphi(u) \leq \varphi(v)$. Now for $u \leq v$ with $u \in \mathcal{O}$ and $v \in \mathbf{C} \setminus \{0\}$ we have $\varphi(u) \leq \varphi(v) = \sup\{\varphi(t) \mid t \in \mathcal{O} \cap [0,v)\}$ since $u \in \mathcal{O} \cap [0,v)$. For $u \leq v$ with $u \in \mathbf{C} \setminus \{0\}$ and $v \in \mathcal{O}$ we have $\varphi(u) = \sup\{\varphi(t) \mid t \in \mathcal{O} \cap [0,u)\} \leq \varphi(v)$ since $\varphi(t) \leq \varphi(v)$ for all $t \in \mathcal{O} \cap [0,u)$, because for such t we have $t < u \leq v$ and φ is increasing on $\mathcal{O}$.

Proposition 2.20

Proposition 2.20. The Cantor-Lebesgue function φ is an increasing continuous function that maps $[0,1]$ onto $[0,1]$. Its derivative exists on the open set $\mathcal{O} = [0,1] \setminus \mathbf{C}$ and $\varphi'(x) = 0$ for $x \in \mathcal{O}$.

Proof. Since φ is increasing on $\mathcal{O}$, then for any $u, v \in \mathcal{O}$ with $u \leq v$ we have $\varphi(u) \leq \varphi(v)$. Now for $u \leq v$ with $u \in \mathcal{O}$ and $v \in \mathbf{C} \setminus \{0\}$ we have $\varphi(u) \leq \varphi(v) = \sup\{\varphi(t) \mid t \in \mathcal{O} \cap [0,v)\}$ since $u \in \mathcal{O} \cap [0,v)$. For $u \leq v$ with $u \in \mathbf{C} \setminus \{0\}$ and $v \in \mathcal{O}$ we have
$\varphi(u) = \sup\{\varphi(t) \mid t \in \mathcal{O} \cap [0,u)\} \leq \varphi(v)$ since $\varphi(t) \leq \varphi(v)$ for all $t \in \mathcal{O} \cap [0,u)$, because for such t we have $t < u \leq v$ and φ is increasing on $\mathcal{O}$. For $u < v$ with $u, v \in \mathbf{C} \setminus \{0\}$ we have that there is some $w \in \mathcal{O}$ with $u < w < v$ and so $\varphi(u) \leq \varphi(w) \leq \varphi(v)$ by the above arguments. Therefore, φ is an increasing function on $[0,1]$.

Proposition 2.20

Proposition 2.20. The Cantor-Lebesgue function φ is an increasing continuous function that maps $[0,1]$ onto $[0,1]$. Its derivative exists on the open set $\mathcal{O} = [0,1] \setminus \mathbf{C}$ and $\varphi'(x) = 0$ for $x \in \mathcal{O}$.

Proof. Since φ is increasing on $\mathcal{O}$, then for any $u, v \in \mathcal{O}$ with $u \leq v$ we have $\varphi(u) \leq \varphi(v)$. Now for $u \leq v$ with $u \in \mathcal{O}$ and $v \in \mathbf{C} \setminus \{0\}$ we have $\varphi(u) \leq \varphi(v) = \sup\{\varphi(t) \mid t \in \mathcal{O} \cap [0,v)\}$ since $u \in \mathcal{O} \cap [0,v)$. For $u \leq v$ with $u \in \mathbf{C} \setminus \{0\}$ and $v \in \mathcal{O}$ we have
$\varphi(u) = \sup\{\varphi(t) \mid t \in \mathcal{O} \cap [0,u)\} \leq \varphi(v)$ since $\varphi(t) \leq \varphi(v)$ for all $t \in \mathcal{O} \cap [0,u)$, because for such t we have $t < u \leq v$ and φ is increasing on $\mathcal{O}$. For $u < v$ with $u, v \in \mathbf{C} \setminus \{0\}$ we have that there is some $w \in \mathcal{O}$ with $u < w < v$ and so $\varphi(u) \leq \varphi(w) \leq \varphi(v)$ by the above arguments. Therefore, φ is an increasing function on $[0,1]$.

Next, continuity...

Proposition 2.20

Proposition 2.20. The Cantor-Lebesgue function φ is an increasing continuous function that maps $[0,1]$ onto $[0,1]$. Its derivative exists on the open set $\mathcal{O} = [0,1] \setminus \mathbf{C}$ and $\varphi'(x) = 0$ for $x \in \mathcal{O}$.

Proof. Since φ is increasing on $\mathcal{O}$, then for any $u, v \in \mathcal{O}$ with $u \leq v$ we have $\varphi(u) \leq \varphi(v)$. Now for $u \leq v$ with $u \in \mathcal{O}$ and $v \in \mathbf{C} \setminus \{0\}$ we have $\varphi(u) \leq \varphi(v) = \sup\{\varphi(t) \mid t \in \mathcal{O} \cap [0,v)\}$ since $u \in \mathcal{O} \cap [0,v)$. For $u \leq v$ with $u \in \mathbf{C} \setminus \{0\}$ and $v \in \mathcal{O}$ we have
$\varphi(u) = \sup\{\varphi(t) \mid t \in \mathcal{O} \cap [0,u)\} \leq \varphi(v)$ since $\varphi(t) \leq \varphi(v)$ for all $t \in \mathcal{O} \cap [0,u)$, because for such t we have $t < u \leq v$ and φ is increasing on $\mathcal{O}$. For $u < v$ with $u, v \in \mathbf{C} \setminus \{0\}$ we have that there is some $w \in \mathcal{O}$ with $u < w < v$ and so $\varphi(u) \leq \varphi(w) \leq \varphi(v)$ by the above arguments. Therefore, φ is an increasing function on $[0,1]$.
Next, continuity…

Proposition 2.20 (continued 1)

Proof (continued). φ is continuous at each point of $\mathcal{O}$ since φ is constant on each open interval component of $\mathcal{O}$. Now consider $x_0 \in \mathbf{C}$ with $x_0 \neq 0, 1$. For $k \in \mathbb{N}$ sufficiently large, we have that x_0 lies between two consecutive open intervals in $[0,1] \setminus C_k$. Let a_k lie in the lower of these two components and b_k lie in the upper of these two components. Function φ is defined to increase by $1/2^k$ across consecutive intervals in $[0,1] \setminus C_k$, therefore $a_k < x_0 < b_k$ and $\varphi(b_k) - \varphi(a_k) = 1/2^k$. Now k can be arbitrarily large and such open interval components exist, so for given $\varepsilon > 0$ if $k \in \mathbb{N}$ is chosen such that $\varepsilon < 1/2^k$ then for $\delta > 0$ such that $\delta < \min\{x_0 - a_k, b_k - x_0 \mid a_k$ and b_k are as described above as elements of $[0,1] \setminus C_k\}$ then we have $|x_0 - x| < \delta$ implies $|\varphi(x_0) - \varphi(x)| \leq \varphi(b_k) - \varphi(a_k) = 1/2^k < \varepsilon$ (we are using the fact that φ is increasing here) and so φ is continuous at $x_0 \in \mathbf{C} \setminus \{0,1\}$. Next, φ takes on the value $1/2^k$ for $x \in \mathbf{C}$ and x "near" 0 and so φ is continuous at $x_0 = 0$ where $\varphi(0) = 0$;

Proposition 2.20 (continued 1)

Proof (continued). φ is continuous at each point of $\mathcal{O}$ since φ is constant on each open interval component of $\mathcal{O}$. Now consider $x_0 \in \mathbf{C}$ with $x_0 \neq 0, 1$. For $k \in \mathbb{N}$ sufficiently large, we have that x_0 lies between two consecutive open intervals in $[0,1] \setminus C_k$. Let a_k lie in the lower of these two components and b_k lie in the upper of these two components. Function φ is defined to increase by $1/2^k$ across consecutive intervals in $[0,1] \setminus C_k$, therefore $a_k < x_0 < b_k$ and $\varphi(b_k) - \varphi(a_k) = 1/2^k$. Now k can be arbitrarily large and such open interval components exist, so for given $\varepsilon > 0$ if $k \in \mathbb{N}$ is chosen such that $\varepsilon < 1/2^k$ then for $\delta > 0$ such that $\delta < \min\{x_0 - a_k, b_k - x_0 \mid a_k$ and b_k are as described above as elements of $[0,1] \setminus C_k\}$ then we have $|x_0 - x| < \delta$ implies $|\varphi(x_0) - \varphi(x)| \leq \varphi(b_k) - \varphi(a_k) = 1/2^k < \varepsilon$ (we are using the fact that φ is increasing here) and so φ is continuous at $x_0 \in \mathbf{C} \setminus \{0, 1\}$. Next, φ takes on the value $1/2^k$ for $x \in \mathbf{C}$ and x "near" 0 and so φ is continuous at $x_0 = 0$ where $\varphi(0) = 0$; if $x_0 = 1$ then we know that φ takes on the values $1 - 1/2^k$ for $x \in \mathbf{C}$ and x "near" 1 and so f is continuous at $x_0 = 1$ where $\varphi(1) = 1$.

Proposition 2.20 (continued 1)

Proof (continued). φ is continuous at each point of $\mathcal{O}$ since φ is constant on each open interval component of $\mathcal{O}$. Now consider $x_0 \in \mathbf{C}$ with $x_0 \neq 0, 1$. For $k \in \mathbb{N}$ sufficiently large, we have that x_0 lies between two consecutive open intervals in $[0,1] \setminus C_k$. Let a_k lie in the lower of these two components and b_k lie in the upper of these two components. Function φ is defined to increase by $1/2^k$ across consecutive intervals in $[0,1] \setminus C_k$, therefore $a_k < x_0 < b_k$ and $\varphi(b_k) - \varphi(a_k) = 1/2^k$. Now k can be arbitrarily large and such open interval components exist, so for given $\varepsilon > 0$ if $k \in \mathbb{N}$ is chosen such that $\varepsilon < 1/2^k$ then for $\delta > 0$ such that $\delta < \min\{x_0 - a_k, b_k - x_0 \mid a_k$ and b_k are as described above as elements of $[0,1] \setminus C_k\}$ then we have $|x_0 - x| < \delta$ implies $|\varphi(x_0) - \varphi(x)| \leq \varphi(b_k) - \varphi(a_k) = 1/2^k < \varepsilon$ (we are using the fact that φ is increasing here) and so φ is continuous at $x_0 \in \mathbf{C} \setminus \{0, 1\}$. Next, φ takes on the value $1/2^k$ for $x \in \mathbf{C}$ and x "near" 0 and so φ is continuous at $x_0 = 0$ where $\varphi(0) = 0$; if $x_0 = 1$ then we know that φ takes on the values $1 - 1/2^k$ for $x \in \mathbf{C}$ and x "near" 1 and so f is continuous at $x_0 = 1$ where $\varphi(1) = 1$.

Proposition 2.20 (continued 2)

Proposition 2.20. The Cantor-Lebesgue function φ is an increasing continuous function that maps $[0,1]$ onto $[0,1]$. Its derivative exists on the open set $\mathcal{O} = [0,1] \setminus \mathbf{C}$ and $\varphi'(x) = 0$ for $x \in \mathcal{O}$.

Proof (continued). Since φ is constant on each of the open intervals in $\mathcal{O}$, then $\varphi'(x) = 0$ for all $x \in \mathcal{O}$. Since $\mathbf{C}$ has measure zero by Proposition 2.19 and so by the Excision Property (Lemma 2.4.A) $m(\mathcal{O}) = m([0,1] \setminus \mathbf{C}) = m([0,1]) - m(\mathbf{C}) = 1$.

Finally, since $\varphi(0) = 0$, $\varphi(1) = 1$, φ is increasing, and φ is continuous, then by the Intermediate Value Theorem, φ maps $[0,1]$ onto $[0,1]$. □

Proposition 2.20 (continued 2)

Proposition 2.20. The Cantor-Lebesgue function φ is an increasing continuous function that maps $[0,1]$ onto $[0,1]$. Its derivative exists on the open set $\mathcal{O} = [0,1] \setminus \mathbf{C}$ and $\varphi'(x) = 0$ for $x \in \mathcal{O}$.

Proof (continued). Since φ is constant on each of the open intervals in $\mathcal{O}$, then $\varphi'(x) = 0$ for all $x \in \mathcal{O}$. Since $\mathbf{C}$ has measure zero by Proposition 2.19 and so by the Excision Property (Lemma 2.4.A)
$m(\mathcal{O}) = m([0,1] \setminus \mathbf{C}) = m([0,1]) - m(\mathbf{C}) = 1.$

Finally, since $\varphi(0) = 0$, $\varphi(1) = 1$, φ is increasing, and φ is continuous, then by the Intermediate Value Theorem, φ maps $[0,1]$ onto $[0,1]$. □

Proposition 2.21

Proposition 2.21. Let φ be the Cantor-Lebesgue function and define the function ψ on $[0,1]$ by $\psi(x) = \varphi(x) + x$. Then ψ is a strictly increasing continuous function that maps $[0,1]$ onto $[0,2]$,

(i) maps the Cantor set **C** onto a measurable set of positive measure and

(ii) maps a measurable set, a subset of the Cantor set, onto a nonmeasurable set.

Proof. Since ψ is the sum of two continuous increasing functions, one of which is strictly increasing, then ψ is continuous and strictly increasing. Since $\psi(0) = 0$ and $\psi(1) = 2$ then $\psi([0,1]) = [0,2]$.

Proposition 2.21

Proposition 2.21. Let φ be the Cantor-Lebesgue function and define the function ψ on $[0,1]$ by $\psi(x) = \varphi(x) + x$. Then ψ is a strictly increasing continuous function that maps $[0,1]$ onto $[0,2]$,

(i) maps the Cantor set $\mathbf{C}$ onto a measurable set of positive measure and

(ii) maps a measurable set, a subset of the Cantor set, onto a nonmeasurable set.

Proof. Since ψ is the sum of two continuous increasing functions, one of which is strictly increasing, then ψ is continuous and strictly increasing. Since $\psi(0) = 0$ and $\psi(1) = 2$ then $\psi([0,1]) = [0,2]$. Now $[0,1] = \mathbf{C} \cup \mathcal{O}$ and since ψ is one to one then $[0,2] = \psi(\mathbf{C}) \cup \psi(\mathcal{O})$. Now a strictly increasing continuous function defined on an interval has a continuous inverse (see Theorem 4-16 of my Analysis 1 [MATH 4217/5217] notes on 4.2. Monotone and Inverse Functions). Therefore $\psi(\mathbf{C})$ is closed and $\psi(\mathcal{O})$ is open (inverse images of open/closed sets under a continuous function is open/closed; see Proposition 1.22) and so both are measurable.

Proposition 2.21

Proposition 2.21. Let φ be the Cantor-Lebesgue function and define the function ψ on $[0, 1]$ by $\psi(x) = \varphi(x) + x$. Then ψ is a strictly increasing continuous function that maps $[0, 1]$ onto $[0, 2]$,

(i) maps the Cantor set $\mathbf{C}$ onto a measurable set of positive measure and

(ii) maps a measurable set, a subset of the Cantor set, onto a nonmeasurable set.

Proof. Since ψ is the sum of two continuous increasing functions, one of which is strictly increasing, then ψ is continuous and strictly increasing. Since $\psi(0) = 0$ and $\psi(1) = 2$ then $\psi([0, 1]) = [0, 2]$. Now $[0, 1] = \mathbf{C} \mathbin{\dot\cup} \mathcal{O}$ and since ψ is one to one then $[0, 2] = \psi(\mathbf{C}) \mathbin{\dot\cup} \psi(\mathcal{O})$. Now a strictly increasing continuous function defined on an interval has a continuous inverse (see Theorem 4-16 of my Analysis 1 [MATH 4217/5217] notes on 4.2. Monotone and Inverse Functions). Therefore $\psi(\mathbf{C})$ is closed and $\psi(\mathcal{O})$ is open (inverse images of open/closed sets under a continuous function is open/closed; see Proposition 1.22) and so both are measurable.

Proposition 2.21 (continued 1)

Proof (continued). Let $\mathcal{O} = \dot\cup_{k=1}^{\infty} I_k$ where the I_k are the connected components of $\mathcal{O}$. Then φ is constant on each I_k and so ψ maps I_k onto a translated copy of itself (translated by the constant given by φ on I_k) of the same length (the "$+x$" part of ψ is the identity function). Since ψ is one to one, the collection $\{\psi(I_k)\}_{k=1}^{\infty}$ is disjoint. By countable additivity (Proposition 2.13),

$$m(\psi(\mathcal{O})) = m(\psi(\dot\cup_{k=1}^{\infty} I_k)) = m(\dot\cup_{k=1}^{\infty} \psi(I_k)) = \sum_{k=1}^{\infty} m(\psi(I_k))$$

$$= \sum_{k=1}^{\infty} \ell(\psi(I_k)) = \sum_{k=1}^{\infty} \ell(I_k) = \sum_{k=1}^{\infty} m(I_k) = m(\dot\cup_{k=1}^{\infty} I_k) = m(\mathcal{O}).$$

Proposition 2.21 (continued 1)

Proof (continued). Let $\mathcal{O} = \dot\cup_{k=1}^{\infty} I_k$ where the I_k are the connected components of $\mathcal{O}$. Then φ is constant on each I_k and so ψ maps I_k onto a translated copy of itself (translated by the constant given by φ on I_k) of the same length (the "$+x$" part of ψ is the identity function). Since ψ is one to one, the collection $\{\psi(I_k)\}_{k=1}^{\infty}$ is disjoint. By countable additivity (Proposition 2.13),

$$m(\psi(\mathcal{O})) = m(\psi(\dot\cup_{k=1}^{\infty} I_k)) = m(\dot\cup_{k=1}^{\infty} \psi(I_k)) = \sum_{k=1}^{\infty} m(\psi(I_k))$$

$$= \sum_{k=1}^{\infty} \ell(\psi(I_k)) = \sum_{k=1}^{\infty} \ell(I_k) = \sum_{k=1}^{\infty} m(I_k) = m(\dot\cup_{k=1}^{\infty} I_k) = m(\mathcal{O}).$$

But $m(\mathbf{C}) = 0$ and $m(\mathcal{O}) = 1$, so $m(\psi(\mathcal{O})) = 1$. Hence, since $[0, 2] = \psi(\mathcal{O}) \dot\cup \psi(\mathbf{C})$, then $m(\psi(\mathbf{C})) = 1$ and (i) follows.

Proposition 2.21 (continued 1)

Proof (continued). Let $\mathcal{O} = \cup_{k=1}^{\infty} I_k$ where the I_k are the connected components of $\mathcal{O}$. Then φ is constant on each I_k and so ψ maps I_k onto a translated copy of itself (translated by the constant given by φ on I_k) of the same length (the "$+x$" part of ψ is the identity function). Since ψ is one to one, the collection $\{\psi(I_k)\}_{k=1}^{\infty}$ is disjoint. By countable additivity (Proposition 2.13),

$$m(\psi(\mathcal{O})) = m(\psi(\cup_{k=1}^{\infty} I_k)) = m(\cup_{k=1}^{\infty} \psi(I_k)) = \sum_{k=1}^{\infty} m(\psi(I_k))$$

$$= \sum_{k=1}^{\infty} \ell(\psi(I_k)) = \sum_{k=1}^{\infty} \ell(I_k) = \sum_{k=1}^{\infty} m(I_k) = m(\cup_{k=1}^{\infty} I_k) = m(\mathcal{O}).$$

But $m(\mathbf{C}) = 0$ and $m(\mathcal{O}) = 1$, so $m(\psi(\mathcal{O})) = 1$. Hence, since $[0,2] = \psi(\mathcal{O}) \cup \psi(\mathbf{C})$, then $m(\psi(\mathbf{C})) = 1$ and (i) follows.

Proposition 2.21 (continued 2)

Proposition 2.21. Let φ be the Cantor-Lebesgue function and define the function ψ on $[0, 1]$ by $\psi(x) = \varphi(x) + x$. Then ψ is a strictly increasing continuous function that maps $[0, 1]$ onto $[0, 2]$,

(i) maps the Cantor set $\mathbf{C}$ onto a measurable set of positive measure and

(ii) maps a measurable set, a subset of the Cantor set, onto a nonmeasurable set.

Proof (continued). To verify (ii), notice that Vitali's Construction of a Nonmeasurable Set (Theorem 2.17) implies that $\psi(\mathbf{C})$ contains a nonmeasurable subset W. The set $\psi^{-1}(W)$ is measurable by Proposition 2.4 since $\psi^{-1}(W) \subset \mathbf{C}$ and $\mathbf{C}$ has measure 0 (so by monotonicity $\psi^{-1}(W)$ has measure 0). So $\psi^{-1}(W)$ is a measurable subset of $\mathbf{C}$ which is mapped by ψ onto a nonmeasurable set. □

Proposition 2.21 (continued 2)

Proposition 2.21. Let φ be the Cantor-Lebesgue function and define the function ψ on $[0,1]$ by $\psi(x) = \varphi(x) + x$. Then ψ is a strictly increasing continuous function that maps $[0,1]$ onto $[0,2]$,

(i) maps the Cantor set $\mathbf{C}$ onto a measurable set of positive measure and

(ii) maps a measurable set, a subset of the Cantor set, onto a nonmeasurable set.

Proof (continued). To verify (ii), notice that Vitali's Construction of a Nonmeasurable Set (Theorem 2.17) implies that $\psi(\mathbf{C})$ contains a nonmeasurable subset W. The set $\psi^{-1}(W)$ is measurable by Proposition 2.4 since $\psi^{-1}(W) \subset \mathbf{C}$ and $\mathbf{C}$ has measure 0 (so by monotonicity $\psi^{-1}(W)$ has measure 0). So $\psi^{-1}(W)$ is a measurable subset of $\mathbf{C}$ which is mapped by ψ onto a nonmeasurable set. □

Proposition 2.22

Proposition 2.22. There is a measurable set, a subset of the Cantor set, that is not a Borel set.

Proof. The strictly increasing continuous function ψ of Proposition 2.21 maps a measurable set A ($A = \psi^{-1}(W)$ in the notation of the proof of Proposition 2.21) onto a nonmeasurable set B ($B = W$ in the proof of Proposition 2.21).

Proposition 2.22

Proposition 2.22. There is a measurable set, a subset of the Cantor set, that is not a Borel set.

Proof. The strictly increasing continuous function ψ of Proposition 2.21 maps a measurable set A ($A = \psi^{-1}(W)$ in the notation of the proof of Proposition 2.21) onto a nonmeasurable set B ($B = W$ in the proof of Proposition 2.21). By Exercise 2.47, a continuous strictly increasing function that is defined on an interval maps Borel sets to Borel sets. So set A is not Borel, or else $B = \psi(A)$ would be Borel and so measurable. □

Proposition 2.22

Proposition 2.22. There is a measurable set, a subset of the Cantor set, that is not a Borel set.

Proof. The strictly increasing continuous function ψ of Proposition 2.21 maps a measurable set A ($A = \psi^{-1}(W)$ in the notation of the proof of Proposition 2.21) onto a nonmeasurable set B ($B = W$ in the proof of Proposition 2.21). By Exercise 2.47, a continuous strictly increasing function that is defined on an interval maps Borel sets to Borel sets. So set A is not Borel, or else $B = \psi(A)$ would be Borel and so measurable. □