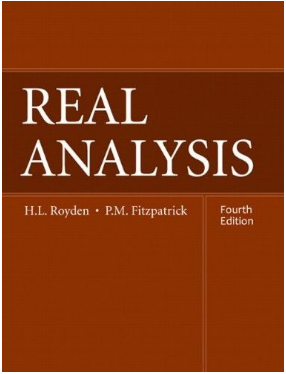
Real Analysis

Chapter 20. The Construction of Particular Measures

20.1. Product Measures: Fubini and Tonelli Theorems—Proofs



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Lemma 20.1. Let $\{A_k \times B_k\}_{k=1}^{\infty}$ be a countable disjoint collection of measurable rectangles whose union also is a measurable rectangle $A \times B$. Notice that index k ranges over ALL of the rectangles which compose $A \times B$ (so there is no $A_i \times B_+ j$ where $i \neq j$). Then

$$\mu(A) \cdot \nu(B) = \sum_{k=1}^{\infty} \mu(A_k) \cdot \nu(B_k).$$

Proof. Fix a point $x \in A$. For each $y \in B$, the point $(x, y) \in A \times B$ and since $\{A_k \times B_k\}_{k=1}^{\infty}$ is a disjoint collection, then (x, y) is in exactly one $A_k \times B_k$.

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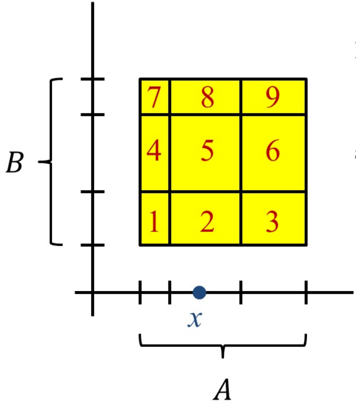

Here,

$$B = \bigcup_{\{k \mid x \in A_k\}} B_k = B_2 \cup B_5 \cup B_8$$

and so

$$\nu(B) = \sum_{x \in A_k} \nu(B_k)\chi_{A_k}(x)$$
$$= \nu(B_2) + \nu(B_5) + \nu(B_8).$$

Equivalently, we have $\nu(B) = \sum_{x \in A_k} \nu(B)\chi_{A_k}(x)$ for $x \in A$. So both for $x \in A$ and $x \notin A$ we have $\nu(B)\chi_A(x) = \sum_{k=1}^{\infty} \nu(B_k)\chi_{A_k}(x)$ for all $x \in X$.

Lemma 20.1 (continued 2)

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Proof (continued). Now $\sum_{k=1}^{\infty} \nu(B_k)\chi_{A_k}(x)$ has partial sums which form a monotone increasing sequence of nonnegative functions. So by the Monotone Convergence Theorem for general measurable spaces (see page 370), $\int_X \nu(B)\chi_A(x)\,d\mu = \nu(B)\mu(A) = \int_X \left(\sum_{k=1}^{\infty} \nu(B_k)\chi_A\right) d\mu =$

$$\sum_{k=1}^{\infty}\left(\int_X \nu(B_k)\chi_A(x)\,d\mu\right) = \sum_{k=1}^{\infty} \nu(B_k)\mu(A_k). \qquad \square$$

Lemma 20.1 (continued 2)

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□

Proposition 20.2

Proposition 20.2. Let $\mathcal{R}$ be the collection of measurable rectangles in $X \times Y$ and for a measurable rectangle $A \times B$ define $\lambda(A \times B) = \mu(A) \cdot \nu(B)$. Then $\mathcal{R}$ is a semiring and $\lambda : \mathcal{R} \to [0, \infty]$ is a premeasure.

Proof. To show $\mathcal{R}$ is a semiring, we need to show that it is closed under finite intersections and that relative complements of $\mathcal{R}$ are finite disjoint unions of elements of $\mathcal{R}$. Let $A_1 \times B_1$ and $A_2 \times B_2$ be measurable rectangles.

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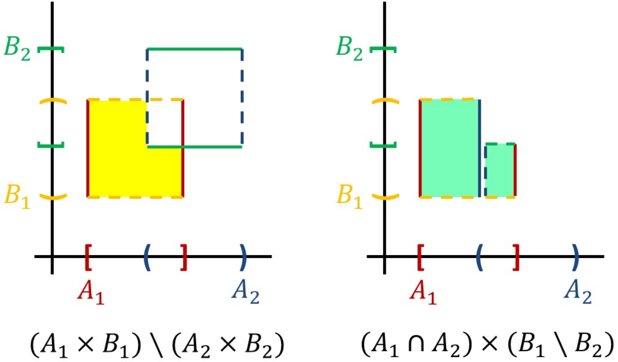
Proposition 20.2 (continued 1)

Proof (continued). Next,
$(A_1 \times B_1) \setminus (A_2 \times B_2) = [(A_1 \setminus A_2) \times B_1] \,\dot\cup\, [(A_1 \cap A_2) \times (B_1 \setminus B_2)]$.



So the relative complement of elements of $\mathcal{R}$ is the union of two disjoint elements of $\mathcal{R}$ (again, we have $\mathcal{A}$ and $\mathcal{B}$ are σ-algebras. so $A_1 \setminus A_2 \in \mathcal{A}$, $A_1 \cup A_2 \in \mathcal{A}$, $B_1 \setminus B_2 \in \mathcal{B}$). So $\mathcal{R}$ is a semiring.

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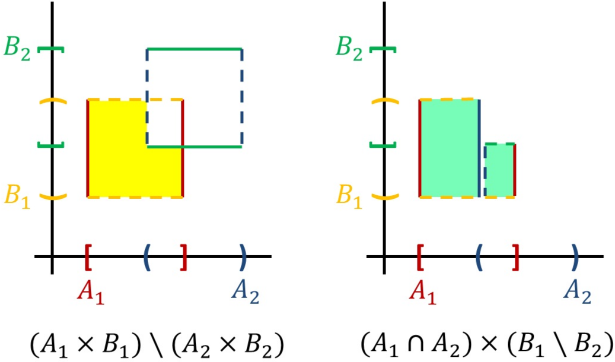

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Proof (continued). To show λ is a premeasure, we must show that λ is finite additive and countably monotone (and that $\lambda(\varnothing) = 0$ which follows from the fact that $\lambda(\varnothing) = \lambda(\varnothing \times \varnothing) = \mu(\varnothing)\nu(\varnothing = 0)$. Lemma 20.1 gives countable additivity. For countable monotonicity. let $E \in \mathcal{R}$ be covered by $\{E_k\}_{k=1}^{\infty} \subset \mathcal{R}$. Since $\mathcal{R}$ is a semiring we can assume WLOG that the E_k are disjoint and that $E = \dot{\cup}_{k=1}^{\infty}(E \cap E_k)$ where each $E \cap E_k$ is a measurable rectangle.

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$$\begin{aligned}\lambda(E) &= \sum_{k=1}^{\infty} \lambda(E \cap E_k) \text{ by Lemma 20.1} \\ &\leq \sum_{k=1}^{\infty} \lambda(E_k) \text{ since } \lambda \text{ is monotone (because } \mu \text{ and } \nu \text{ are monotone).}\end{aligned}$$

So λ is countable monotone and λ is a premeasure. □

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Lemma 20.3

Lemma 20.3. Let $E \subset X \times Y$ be an $\mathcal{R}_{\sigma\delta}$ set for which $(\mu \times \nu)(E) < \infty$. Then for all $x \in X$, the x-section of set E, E_x, is a ν-measurable subset of Y, the function $x \mapsto \nu(E_x)$ for $x \in X$ is a μ-measurable function and

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$$(\mu \times \nu)(E) = \mu(A)\nu(B) = \nu(B) \int_X \chi_A(x)\, d\mu(x) = \int_X \nu(E_x)\, d\mu(x),$$

so the result holds for E a measurable rectangle.

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Proof (continued). (2) Suppose E in an $\mathcal{R}_\sigma$ set. Since $\mathcal{R}$ is a semiring, there is a disjoint collection of measurable rectangles $\{A_k \times B_k\}_{k=1}^\infty$ whose union is E. For fixed $x \in X$, we have $E_x = \dot\cup_{k=1}^\infty (A_k \times B_k)_x$. Thus E_x is the countable disjoint union of some of the B_k's (the ones for which $x \in A_k$).

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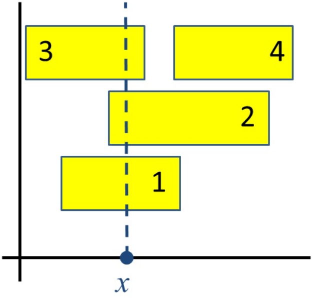

$$E_x = B_1 \cup B_2 \cup B_3$$

So by the countable additivity of ν (by the definition of measure), $\nu(E_x) = \sum_{k=1}^\infty \nu(A_k \times B_k)_x)$.

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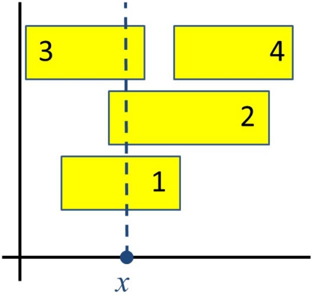

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Proof (continued). So we have

$$\begin{aligned}
\int_X \nu(E_x)\,d\mu(x) &= \int_X \left(\sum_{k=1}^{\infty} \nu((A_k \times B_k)_x)\right) d\mu(x) \\
&= \sum_{k=1}^{\infty} \left(\int_X \nu((A_k \times B_k)_x)\,d\mu(x)\right) \text{ by the Monotone} \\
&\quad \text{Convergence Theorem; the partial sums are increasing} \\
&= \sum_{k=1}^{\infty} \mu(A_k)\nu(B_k) \text{ by part (1),} \\
&\quad \text{since } A_k \times B_k \text{ is a measurable rectangle} \\
&= (\mu \times \nu)(E) \text{ by the definition of } \mu \times \nu
\end{aligned}$$

So the result holds for $E \in \mathcal{R}_\sigma$.

Lemma 20.3 (continued 3)

Proof (continued). (3) Suppose E is in $\mathcal{R}_{\sigma\delta}$ and $(\mu \times \nu)(E) < \infty$. Since $\mathcal{R}$ is a semiring (closed under finite intersections), there is a descending sequence $\{E_k\}_{k=1}^{\infty}$ of sets in $\mathcal{R}_\sigma$ whose intersection is E. Since $(\mu \times \nu)(E) < \infty$, without loss of generality $(\mu \times \nu)(E) < \infty$ (from the definition of $\mu \times \nu$ in terms of the outer measure induced by the premeasure $\mu \times \nu$ on $\mathcal{R}$).

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$$\lim_{k \to \infty} (\mu \times \nu) E_k) = (\mu \times \nu)(E). \qquad (3)$$

Since $E_1 \in \mathcal{R}_\sigma$ and the result holds for $\mathcal{R}_\sigma$ sets, $(\mu \times \nu)(E_1) = \int_X \nu((E_1)_x)\, d\mu(x)$, and since $(\mu \times \nu)(E_1) < \infty$ and $\nu((E_1)_x)$ is nonnegative, then $\nu((E_1)_x) < \infty$ for almost all $x \in X$ by Proposition 18.9.

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Lemma 20.3 (continued 3)

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Lemma 20.3 (continued 4)

Proof (continued). Furthermore, the function $x \mapsto \nu((E_1)_x)$ is nonnegative and integrable (since $(\mu \times \nu)(E_1) < \infty$) and for each $k \in \mathbb{N}$, dominates a.e. the function $x \mapsto \nu((E_k)_x)$ since the E_k form a descending sequence. So we have

$$\begin{aligned} \int_X \nu(E_x)\, d\mu(x) &= \int_X \left(\lim_{k\to\infty} \nu(E_k)_x \right) d\mu(x) \\ &= \lim_{k\to\infty} \left(\int_X \nu((E_k)_x)\, d\mu(x) \right) \text{ by the Lebesgue Dominated Convergence Theorem} \\ &= \lim_{k\to\infty} (\mu \times \nu)(E_k) \text{ since the result holds on } \mathcal{R}_\sigma \text{ sets } E_k \\ &= (\mu \times \nu)(E) \text{ from (3)} \end{aligned}$$

So the result holds for $\mathcal{R}_{\sigma\delta}$ sets. □

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$$\begin{aligned}
\int_X \nu(E_x)\,d\mu(x) &= \int_X \left(\lim_{k\to\infty} \nu(E_k)_x\right) d\mu(x) \\
&= \lim_{k\to\infty} \left(\int_X \nu((E_k)_x)\,d\mu(x)\right) \text{ by the Lebesgue} \\
&\qquad\qquad \text{Dominated Convergence Theorem} \\
&= \lim_{k\to\infty} (\mu \times \nu)(E_k) \text{ since the result holds on } \mathcal{R}_\sigma \text{ sets } E_k \\
&= (\mu \times \nu)(E) \text{ from (3)}
\end{aligned}$$

So the result holds for $\mathcal{R}_{\sigma\delta}$ sets. □

Lemma 20.4

Lemma 20.4. Assume the measure ν is complete. Let$E \subset X \times Y$ be measurable with respect to $\mu \times \nu$. If $(\mu \times \nu)(E) = 0$, then almost all $x \in X$, the x-section of E, E_x, is ν-measurable and $\nu(E_x) = 0$. Therefore

$$0 = (\mu \times \nu)(E) = \int_X \nu(E_x)\, d\mu(x) = 0.$$

Proof. Since $(\mu \times \nu)(E) < \infty$ it follows from Proposition 17.10 that there is a set $A \in \mathcal{R}_{\sigma\delta}$ for which $E \subset A$ and $(\mu \times \nu)(A) = (\mu \times \nu)(E) = 0$.

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Lemma 20.4. Assume the measure ν is complete. Let$E \subset X \times Y$ be measurable with respect to $\mu \times \nu$. If $(\mu \times \nu)(E) = 0$, then almost all $x \in X$, the x-section of E, E_x, is ν-measurable and $\nu(E_x) = 0$. Therefore

$$0 = (\mu \times \nu)(E) = \int_X \nu(E_x)\, d\mu(x) = 0.$$

Proof. Since $(\mu \times \nu)(E) < \infty$ it follows from Proposition 17.10 that there is a set $A \in \mathcal{R}_{\sigma\delta}$ for which $E \subset A$ and $(\mu \times \nu)(A) = (\mu \times \nu)(E) = 0$. Since A is $\mathcal{R}_{\sigma\delta}$, by Lemma 20.3 we have that for all $x \in X$ that the x-section of A, A_x, is ν-measurable and $(\mu \times \nu)(A) = \int_X \nu(A_x)\, d\mu(x)$. So the integral is 0 for almost all $x \in X$ by Problem 18.19.

Lemma 20.4

Lemma 20.4. Assume the measure ν is complete. Let$E \subset X \times Y$ be measurable with respect to $\mu \times \nu$. If $(\mu \times \nu)(E) = 0$, then almost all $x \in X$, the x-section of E, E_x, is ν-measurable and $\nu(E_x) = 0$. Therefore

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$$\int_X \nu(E_x)\, d\mu(x) = 0 = (\mu \times \nu)(E).$$

□

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Proof. Since $(\mu \times \nu)(E) < \infty$ it follows from Proposition 17.10 that there is a set $A \in \mathcal{R}_{\sigma\delta}$ for which $E \subset A$ and $(\mu \times \nu)(A) = (\mu \times \nu)(E) = 0$. Since A is $\mathcal{R}_{\sigma\delta}$, by Lemma 20.3 we have that for all $x \in X$ that the x-section of A, A_x, is ν-measurable and $(\mu \times \nu)(A) = \int_X \nu(A_x)\, d\mu(x)$. So the integral is 0 for almost all $x \in X$ by Problem 18.19. However, for all $x \in X$ we have $E_x \subset A_x$. By the completeness of ν, $\nu(E_x) = 0$ and so E_x is ν-measurable. So

$$\int_X \nu(E_x)\, d\mu(x) = 0 = (\mu \times \nu)(E).$$

□

Proposition 20.5

Proposition 20.5. Assume the measure ν is complete. Let $E \subset X \times Y$ be measurable with respect to $\mu \times \nu$ and $(\mu \times \nu)(E) < \infty$. The for almost all $x \in X$, the x-section of E, E_x, is a ν-measurable subset of Y, the function $x \mapsto \nu(E_x)$ is μ-measurable for all $x \in X$, and

$$(\mu \times \nu)(E) = \int_X \nu(E_x)\, d\mu(x).$$

Proof. Since $(\mu \times \nu)(E) < \infty$ it follows from Proposition 17.10 that there is a set $A \in \mathcal{R}_{\sigma\delta}$ for which $E \subset Q$ and $(\mu \times \nu)(A \setminus E) = 0$. By the excision property of measure $\mu \times \nu$ (Proposition 17.1), we have $(\mu \times \nu)(E) = (\mu \times \nu)(A)$.

Proposition 20.5

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$$\nu(A_x) = \nu(E_x \mathbin{\dot\cup} (A \setminus E)_x) = \nu(E_x) + \nu((A \setminus E)_x).$$

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Proof. Since $(\mu \times \nu)(E) < \infty$ it follows from Proposition 17.10 that there is a set $A \in \mathcal{R}_{\sigma\delta}$ for which $E \subset Q$ and $(\mu \times \nu)(A \setminus E) = 0$. By the excision property of measure $\mu \times \nu$ (Proposition 17.1), we have $(\mu \times \nu)(E) = (\mu \times \nu)(A)$. Since $A \in \mathcal{R}_{\sigma\delta}$ then by Lemma 20.3, A_x is a ν-measurable function. So by the finite additivity of ν (Proposition 17.1)

$$\nu(A_x) = \nu(E_x \,\dot\cup\, (A \setminus E)_x) = \nu(E_x) + \nu((A \setminus E)_x).$$

Proposition 20.5 (continued)

Proof. Since $(\mu \times \nu)(E \setminus A) = 0$, then by Lemma 20.4 (the completeness of ν is used here) for almost all $x \in X$, $(\setminus E)_x$ is ν-measurable and $\nu((A \setminus E)_x) = 0$. So $\nu(A_x) = \nu(E_x)$ for almost all $x \in X$. So

$$
\begin{aligned}
(\mu \times \nu)(E) &= (\mu \times \nu)(A) \text{ by above} \\
&= \int_X \nu(A_z)\, d\mu(x) \text{ by Lemma 20.3 since } A \in \mathcal{R}_{\sigma\delta} \\
&= \int_X \nu(E_x)\, d\mu(x) \text{ since } \nu(A_x) = \nu(E_x) \text{ a.e. on } X.
\end{aligned}
$$

□

Proposition 20.5 (continued)

Proof. Since $(\mu \times \nu)(E \setminus A) = 0$, then by Lemma 20.4 (the completeness of ν is used here) for almost all $x \in X$, $(\setminus E)_x$ is ν-measurable and $\nu((A \setminus E)_x) = 0$. So $\nu(A_x) = \nu(E_x)$ for almost all $x \in X$. So

$$\begin{aligned} (\mu \times \nu)(E) &= (\mu \times \nu)(A) \text{ by above} \\ &= \int_X \nu(A_z)\, d\mu(x) \text{ by Lemma 20.3 since } A \in \mathcal{R}_{\sigma\delta} \\ &= \int_X \nu(E_x)\, d\mu(x) \text{ since } \nu(A_x) = \nu(E_x) \text{ a.e. on } X. \end{aligned}$$

□

Theorem 20.6

Theorem 20.6. Assume measure ν is complete. Let $\phi : X \times Y \to \mathbb{R}$ be a simple function that is integrable over $X \times Y$ with respect to $\mu \times \nu$. Then for almost all $x \in X$, the x-section of ϕ, $\phi(x, \cdot)$, is integrable over Y with respect to ν and

$$\int_{X \times Y} \phi \, d(\mu \times \nu) = \int_X \left[\int_Y \phi(x, y) \, d\nu(y) \right] \, d\mu(x).$$

Proof. First, if χ_E is a characteristic function on a subset E of $X \times Y$ of finite measure (to get integrability), then

$$\begin{aligned} \int_{X \times Y} \chi_E \, d(\mu \times \nu) &= 1(\mu \times \nu)(E) \text{ where } \varphi = 1 \text{ on } E, \text{ by the definition} \\ &\quad \text{of integral of a characteristic function; page 366} \\ &= \int_X \nu((\chi_E)_x) \, d\mu(x) \text{ by Proposition 20.5} \end{aligned}$$

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Theorem 20.6 (continued)

Proof (continued).

$$\int_{X\times Y} \chi_E \, d(\mu\times\nu) = \int_X \left(\int_Y \varphi(x,y)\, d\nu(y) \right) d\mu(x)$$

since $(\chi_E)_x = \chi_{E_x} = \varphi(x,\cdot) = \begin{cases} 1 & \text{if } y \in E_x \\ 0 & \text{if } y \notin E_x \end{cases}$ and so $\nu((\chi_E)_x) = \int_Y \varphi(x,y)\, d\nu(y)$. So the result holds for characteristic functions.

Theorem 20.6 (continued)

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Second, for general simple and integrable φ, φ is a linear combination of characteristic functions and this result then follows by the linearity of integration (Theorem 18.12) as applied to the integral with respect to ν. □

Theorem 20.6 (continued)

Proof (continued).

$$\int_{X\times Y} \chi_E \, d(\mu\times\nu) \;=\; \int_X \left(\int_Y \varphi(x,y)\, d\nu(y)\right) d\mu(x)$$

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Fubini's Theorem

Fubini's Theorem.
Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be measure spaces and let ν be complete. Let f be integrable over $X \times Y$ with respect to the product measure $\mu \times \nu$. Then for almost all $x \in X$, the x-section of f, $f(x, \cdot)(y)$, is integrable over Y with respect to μ and

$$\int_{X \times Y} f \, d(\mu \times \nu) = \int_X \left[\int_Y f(x, y) \, d\nu(y) \right] d\mu(x).$$

Proof. Since integration is linear (Theorem 18.12), we assume f is nonnegative (otherwise, we break f into f^+ and f^- and consider these parts individually). By the Simple Approximation Theorem there is an increasing sequence $\{\varphi\}$ of simple functions that converges pointwise on $X \times Y$ to f and $0 \leq \varphi_k \leq f$ on $X \times T$ for each $k \in \mathbb{N}$. Since f is integrable over $X \times Y$, each φ_k is integrable over $X \times Y$ (by the Integral Comparison Test).

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$$\int_{X \times Y} f \, d(\mu \times \nu) = \int_X \left[\int_Y f(x, y) \, d\nu(y) \right] d\mu(x).$$

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Fubini's Theorem (continued 1)

Proof (continued). Since each φ_k is simple and integrable, then by Theorem 20.6 we have

$$\int_{X\times Y} \varphi_k \, d(\mu \times \nu) = \int_X \left(\int_Y \varphi_k(x,y) \, d\nu(y) \right) \, d\mu(x).$$

Since $\{\varphi_k\}$ is an increasing sequence convergent to f, we can apply the Monotone Convergence Theorem to get

$$\int_{X\times Y} f \, d(\mu\times\nu) = \int_{X\times Y} \left(\lim_{k\to\infty} \varphi_k \right) \, d(\mu\times\nu) = \lim_{k\to\infty} \left(\int_{X\times Y} \varphi_k \, d(\mu \times \nu) \right)$$

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So we are done if we show

$$\lim_{k\to\infty} \int_X \left(\int_Y \varphi_k(x,y)\, d\nu(y) \right) d\mu(x) = \int_X \left(\int_Y f(x,y)\, f\nu(y) \right) d\mu(x). \quad (7)$$

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$$= \lim_{k\to\infty}\int_X \left(\int_Y \varphi_k(x,y)\, d\nu(y)\right) d\mu(x).$$

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Fubini's Theorem (continued 2)

Proof (continued). If we excise from $X \times Y$ a set of $\mu \times \nu$ measure zero, then the left hand side of (7) remains the same by Additivity of Integrals (Theorem 8.12) since the left hand side of (7) equals $\int_{X \times Y} f\, d(\mu \times \nu)$ by above. If we excise from $X \times Y$ a set E of $\mu \times \nu$ measure zero, then by Lemma 20.4 for almost all $x \in X$, $\nu(E_x) = 0$ where E_x is the x-section of the excised set E. But then $\int_{E_x} f(x, \cdot)(y)\, d\nu(y) = 0$ for almost all $x \in X$, by Problem 18.19. So for almost all $x \in X$, $\int_Y f(x,y)\, d\nu(y) = \int_{Y \setminus E_x} f(x,y)\, d\nu(y)$. Since this holds a.e. on X, we see that the right hand side of (7) also remains unchanged by the excision of $\mu \times \nu$ measure zero set E.

Fubini's Theorem (continued 2)

Proof (continued). If we excise from $X \times Y$ a set of $\mu \times \nu$ measure zero, then the left hand side of (7) remains the same by Additivity of Integrals (Theorem 8.12) since the left hand side of (7) equals $\int_{X \times Y} f \, d(\mu \times \nu)$ by above. If we excise from $X \times Y$ a set E of $\mu \times \nu$ measure zero, then by Lemma 20.4 for almost all $x \in X$, $\nu(E_x) = 0$ where E_x is the x-section of the excised set E. But then $\int_{E_x} f(x, \cdot)(y) \, d\nu(y) = 0$ for almost all $x \in X$, by Problem 18.19. So for almost all $x \in X$,
$\int_Y f(x,y) \, d\nu(y) = \int_{Y \setminus E_x} f(x,y) \, d\nu(y)$. Since this holds a.e. on X, we see that the right hand side of (7) also remains unchanged by the excision of $\mu \times \nu$ measure zero set E. So, without loss of generality we may suppose that for all $x \in X$ and for all $k \in \mathbb{N}$, $\varphi_k(x, \cdot)$ is integrable over Y with respect to ν.

Fubini's Theorem (continued 2)

Proof (continued). If we excise from $X \times Y$ a set of $\mu \times \nu$ measure zero, then the left hand side of (7) remains the same by Additivity of Integrals (Theorem 8.12) since the left hand side of (7) equals $\int_{X\times Y} f \, d(\mu \times \nu)$ by above. If we excise from $X \times Y$ a set E of $\mu \times \nu$ measure zero, then by Lemma 20.4 for almost all $x \in X$, $\nu(E_x) = 0$ where E_x is the x-section of the excised set E. But then $\int_{E_x} f(x,\cdot)(y)\, d\nu(y) = 0$ for almost all $x \in X$, by Problem 18.19. So for almost all $x \in X$, $\int_Y f(x,y)\, d\nu(y) = \int_{Y\setminus E_x} f(x,y)\, d\nu(y)$. Since this holds a.e. on X, we see that the right hand side of (7) also remains unchanged by the excision of $\mu \times \nu$ measure zero set E. So, without loss of generality we may suppose that for all $x \in X$ and for all $k \in \mathbb{N}$, $\varphi_k(x,\cdot)$ is integrable over Y with respect to ν.

Fubini's Theorem (continued 3)

Proof (continued). Fix $x \in X$. Then $\{\varphi_k(x, \cdot)\}$ is an increasing sequence of simple ν-measurable functions that converges pointwise on Y to $f(x, \cdot)$. By Theorem 18.6, $f(x, \cdot)$ is a ν-measurable function, and by the Monotone Convergence Theorem,

$$\int_Y f(x, y)\, d\nu(y) = \int_Y \left(\lim_{k\to\infty} \varphi_k(x, y) \right) d\nu(y) = \lim_{k\to\infty} \int_Y \varphi_k(x, y)\, d\nu(y).$$

For each $x \in X$, define $h(x) = \int_Y f(x, y)\, d\nu(y)$ and $h_k(x) = \int_Y \varphi_k(x, y)\, d\nu(y)$. By Theorem 20.6, each h_k is integrable over X with respect to μ. Now $\{h_k\}$ is an increasing nonnegative sequence (since $\{\varphi_k\}$ is an increasing nonnegative sequence) that converges pointwise on X to h.

Fubini's Theorem (continued 3)

Proof (continued). Fix $x \in X$. Then $\{\varphi_k(x,\cdot)\}$ is an increasing sequence of simple ν-measurable functions that converges pointwise on Y to $f(x,\cdot)$. By Theorem 18.6, $f(x,\cdot)$ is a ν-measurable function, and by the Monotone Convergence Theorem,

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For each $x \in X$, define $h(x) = \int_Y f(x,y)\,d\nu(y)$ and $h_k(x) = \int_Y \varphi_k(x,y)\,d\nu(y)$. By Theorem 20.6, each h_k is integrable over X with respect to μ. Now $\{h_k\}$ is an increasing nonnegative sequence (since $\{\varphi_k\}$ is an increasing nonnegative sequence) that converges pointwise on X to h. So by the Monotone Convergence Theorem,

$$\lim_{k\to\infty}\int_X \left(\int_Y \varphi_k(x,y)\,d\nu(y)\right) = \lim_{k\to\infty}\int_X h_k(x)\,d\mu(x)$$

$$= \int_X h(x)\,d\mu(x) = \int_X \left(\int_Y f(x,y)\,d\nu(y)\right)\,d\mu(x).$$

Fubini's Theorem (continued 3)

Proof (continued). Fix $x \in X$. Then $\{\varphi_k(x, \cdot)\}$ is an increasing sequence of simple ν-measurable functions that converges pointwise on Y to $f(x, \cdot)$. By Theorem 18.6, $f(x, \cdot)$ is a ν-measurable function, and by the Monotone Convergence Theorem,

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Fubini's Theorem (continued 4)

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Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be measure spaces and let ν be complete. Let f be integrable over $X \times Y$ with respect to the product measure $\mu \times \nu$. Then for almost all $x \in X$, the x-section of f, $f(x, \cdot)(y)$, is integrable over Y with respect to μ and

$$\int_{X \times Y} f \, d(\mu \times \nu) = \int_X \left[\int_Y f(x, y) \, d\nu(y) \right] \, d\mu(x).$$

Proof (continued). So (7) holds and we now have

$$\int_{X \times Y} f \, d(\mu \times \nu) = \lim_{k \to \infty} \int_{X \times Y} \varphi_k \, d(\mu \times \nu)$$

$$= \lim_{k \to \infty} \int_X \left(\int_Y \varphi_k(x, y) \, d\nu(y) \right) \, d\mu(x) = \int_X \left(\int_Y f(x, y) \, d\nu(y) \right) d\mu(x).$$

□

Tonelli's Theorem

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Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be two σ-finite measure spaces and ν be complete. Let f be a nonnegative $(\mu \times \nu)$-measurable function on $X \times Y$. Then for almost all $x \in X$, the x-section of function f, $f(x, \cdot)$, is ν-measurable and the function defined a.e. on X by

$$x \mapsto (\text{the integral of } f(x,\cdot) \text{ over } X \text{ with respect to } \nu)$$

is μ-measurable. Moreover,

$$\int_{X \times Y} f \, f(\mu \times \nu) = \int_X \left(\int_Y f(x,y)\, d\nu(y) \right) d\mu(x).$$

Proof. By the Simple Approximation Theorem, there is an increasing sequence $\{\varphi_k\}$ of simple functions that converge pointwise on $X \times Y$ to f, and $0 \leq \varphi_k \leq f$ on $X \times Y$ for all $k \in \mathbb{N}$. The product measure $\mu \times \nu$ is σ-finite since both μ and ν are σ-finite.

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is μ-measurable. Moreover,

$$\int_{X \times Y} f\, f(\mu \times \nu) = \int_X \left(\int_Y f(x, y)\, d\nu(y) \right) d\mu(x).$$

Proof. By the Simple Approximation Theorem, there is an increasing sequence $\{\varphi_k\}$ of simple functions that converge pointwise on $X \times Y$ to f, and $0 \leq \varphi_k \leq f$ on $X \times Y$ for all $k \in \mathbb{N}$. The product measure $\mu \times \nu$ is σ-finite since both μ and ν are σ-finite.

Tonelli's Theorem (continued)

Tonelli's Theorem.
Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be two σ-finite measure spaces and ν be complete. Let f be a nonnegative $(\mu \times \nu)$-measurable function on $X \times Y$. Then for almost all $x \in X$, the x-section of function f, $f(x, \cdot)$, is ν-measurable and the function defined a.e. on X by

$$x \mapsto (\text{the integral of } f(x, \cdot) \text{ over } X \text{ with respect to } \nu)$$

is μ-measurable. Moreover,

$$\int_{X \times Y} f \, f(\mu \times \nu) = \int_X \left(\int_Y f(x, y) \, d\nu(y) \right) \, d\mu(x).$$

Proof (continued). So by (i) of the Simple Approximation Theorem, the φ_k can have the additional property that they vanish outside a set of finite measure (and so are integrable). We now apply Theorem 20.6 to each φ_k, as we did neat the beginning of the proof of Fubini. The remainder of the proof is identical to the proof of Fubini's Theorem. □

Corollary 20.7. Tonelli's Corollary

Corollary 20.7. (Tonelli's Corollary).
Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be two σ-finite, complete measure spaces and f a nonnegative $(\mu \times \nu)$-measurable function of $X \times Y$. Then:

(i) For almost all $x \in X$, the x-section of f, $f(x, \cdot)$, is ν-measurable and the function defined almost everywhere on X by

$$x \mapsto (\text{the integral of } f(x, \cdot) \text{ over } X \text{ with respect to } \nu)$$

is μ-measurable, and

(ii) for almost all $y \in Y$, the y-section of f, $f(\cdot, y)$, is μ-measurable and the function defined almost everywhere on Y by

$$y \mapsto (\text{the integral of } f(\cdot, y) \text{ over } Y \text{ with respect to } \mu)$$

is μ-measurable.

Corollary 20.7. Tonelli's Corollary (continued 1)

Corollary 20.7. (Tonelli's Corollary, continued).
Moreover, if

$$\int_X \left(\int_Y f(x,y)\, d\nu(y) \right) d\mu(x) < \infty,$$

then f is integrable over $X \times Y$ with respect to $\mu \times \nu$ and

$$\int_Y \left(\int_X f(x,y)\, d\mu(x) \right) d\nu(y) = \int_{X \times Y} f\, d(\mu \times \nu)$$

$$= \int_X \left(\int_Y f(x,y)\, d\nu(y) \right) d\mu(x).$$

Proof. Since both measure spaces are σ-finite and ν is complete, Tonelli's Theorem implies that the x-section of f is ν-measurable for almost all $x \in X$ and $x \mapsto \int_Y f(x,y)\, d\nu(y)$ is μ-measurable.

Corollary 20.7. Tonelli's Corollary (continued 1)

Corollary 20.7. (Tonelli's Corollary, continued).
Moreover, if

$$\int_X \left(\int_Y f(x,y)\, d\nu(y) \right) d\mu(x) < \infty,$$

then f is integrable over $X \times Y$ with respect to $\mu \times \nu$ and

$$\int_Y \left(\int_X f(x,y)\, d\mu(x) \right) d\nu(y) = \int_{X \times Y} f\, d(\mu \times \nu)$$

$$= \int_X \left(\int_Y f(x,y)\, d\nu(y) \right) d\mu(x).$$

Proof. Since both measure spaces are σ-finite and ν is complete, Tonelli's Theorem implies that the x-section of f is ν-measurable for almost all $x \in X$ and $x \mapsto \int_Y f(x,y)\, d\nu(y)$ is μ-measurable.

Corollary 20.7. Tonelli's Corollary (continued 2)

Proof (continued). Also,
$\int_{X\times Y} f\, f(\mu\times\nu) = \int_X \left(\int_Y f(x,y)\,d\nu(y)\right)\, d\mu(x)$, and so f is integrable over $X\times Y$ with respect to $\mu\times\nu$ by (10). Now applying Fubini's Theorem, since f is integrable over $X\times Y$ with respect to $\mu\times\nu$ and since μ is complete we have

$$\int_{X\times Y} f\, d(\mu\times\nu) = \int_Y \left(\int_X f(x,y)\,d\mu(x)\,d\nu(y)\right).$$

□

Corollary 20.7. Tonelli's Corollary (continued 2)

Proof (continued). Also,
$\int_{X\times Y} f\, f(\mu\times\nu) = \int_X \left(\int_Y f(x,y)\,d\nu(y)\right)\,d\mu(x)$, and so f is integrable over $X\times Y$ with respect to $\mu\times\nu$ by (10). Now applying Fubini's Theorem, since f is integrable over $X\times Y$ with respect to $\mu\times\nu$ and since μ is complete we have

$$\int_{X\times Y} f\,d(\mu\times\nu) = \int_Y \left(\int_X f(x,y)\,d\mu(x)\,d\nu(y)\right).$$

□