Real Analysis

Chapter 20. The Construction of Particular Measures

20.2. Lebesgue Measures on Euclidean Space $\mathbb{R}^n$—Proofs of Theorems

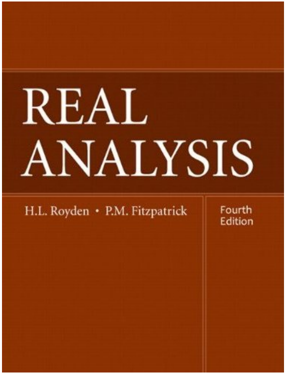

Table of contents

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Lemma 20.8. For each $\varepsilon > 0$, the *ε-dilation* $T_\varepsilon : \mathbb{R}n \to \mathbb{R}^n$ is $T_\varepsilon(x) = \varepsilon n$. Then for each bounded interval I in $\mathbb{R}^n$,

$$\lim_{\varepsilon \to 0} \frac{\mu^{integral}(T_\varepsilon(I))}{\varepsilon^n} = \text{vol}(I).$$

Proof. For bounded interval I in $\mathbb{R}$ with end-points a and b then, by Exercise 20.18,

$$(b-a) - 1 \leq \mu^{integral}(I) \leq (b-a) + 1.$$

So for $I = I_1 \times I_2 \times \cdots \times I_n$ we have

$$\mu^{integral}(I) = \mu^{integral}(I_1)\mu^{integral}(I_2) \cdots \mu^{integral}(I_n)$$

and with I_k having end points a_k and b_k, this implies

$$((b_1 - a_1) - 1)((b_2 - a_2) - 1) \cdots ((b_n - a_n) - 1) \leq \mu^{integral}(I)$$
$$\leq ((b_1 - a_1) + 1)((b_2 - a_2) + 1) \cdots ((b_n - a_n) + 1).$$

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$$\leq ((b_1 - a_1) + 1)((b_2 - a_2) + 1) \cdots ((b_n - a_n) + 1).$$

Lemma 20.8 (continued)

Proof (continued). Since T_ε maps interval I_k to an interval with endpoints εa_k and εb_k, then

$$(\varepsilon(b_1 - a_1) - 1)(\varepsilon(b_2 - a_2) - 1) \cdots (\varepsilon(b_n - a_n) - 1) \leq \mu^{integral}(T_\varepsilon(I))$$

$$\leq (\varepsilon(b_1 - a_1) + 1)(\varepsilon(b_2 - a_2) + 1) \cdots (\varepsilon(b_n - a_n) + 1).$$

Dividing this by ε^n and letting $\varepsilon \to \infty$ we get

$$(b_1 - a_1)(b_2 - a_2) \cdots (b_n - a_n) \leq \lim)_{\varepsilon \to \infty} \frac{\mu^{integral}(T_\varepsilon(I))}{\varepsilon^n}$$

$$\leq (b_1 - a_1)(b_2 - a_2) \cdots (b_n - a_n),$$

or $\text{vol}(I) = \lim_{\varepsilon \to \infty} \frac{\mu^{integral}(T_\varepsilon(I))}{\varepsilon^n}$, as claimed. □

Lemma 20.8 (continued)

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Proposition 20.10

Proposition 20.10. The set function volume, vol : $\mathcal{I} \to [0, \infty)$, is a premeasure on the semiring $\mathcal{I}$ of bounded intervals in $\mathbb{R}^n$.

Proof. By definition of "premeasure," we need to show that vol is finitely additive and countably monotone on the semiring consists only of bounded intervals and the intersection of two bounded intervals is a bounded interval.

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Let I be a bounded interval in $\mathbb{R}^n$ that is the union of the finite disjoint collection of bounded intervals $\{I^k\}_{k=1}^m$ (think of the I^k as disjoint blocks which pack together to produce box I). Then for each $\varepsilon > 0$, the bounded interval $T_\varepsilon(I)$ is the union of the finite disjoint collection of bounded intervals $\{T_\varepsilon(I^k)\}_{k=1}^m$.

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Let I be a bounded interval in $\mathbb{R}^n$ that is the union of the finite disjoint collection of bounded intervals $\{I^k\}_{k=1}^m$ (think of the I^k as disjoint blocks which pack together to produce box I). Then for each $\varepsilon > 0$, the bounded interval $T_\varepsilon(I)$ is the union of the finite disjoint collection of bounded intervals $\{T_\varepsilon(I^k)\}_{k=1}^m$. Since $\mu^{integral}$ is "clearly" finitely additive, then for all $\varepsilon > 0$,

$$\mu^{integral}(T_\varepsilon(I)) = \sum_{k=1}^m \mu^{integral}(T_\varepsilon(I^k)).$$

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$$\mu^{integral}(T_\varepsilon(I)) = \sum_{k=1}^{m} \mu^{integral}(T_\varepsilon(I^k)).$$

Proposition 20.10 (continued 1)

Proof (continued). Dividing both sides of this by ε^n and letting $\varepsilon \to 0$ we get, by Lemma 20.8,

$$\text{vol}(I) = \lim_{\varepsilon \to 0} \frac{\mu^{integral}(T_\varepsilon(I))}{\varepsilon^n} = \lim_{\varepsilon \to 0} \sum_{k=1}^{m} \frac{\mu^{integral}(T_\varepsilon(I^k))}{\varepsilon^n} = \sum_{k=1}^{n} \text{vol}(I^k).$$

Therefore, vol is finitely additive.

For countable monotonicity, let I be a bounded interval in $\mathbb{R}^n$ that is covered by the countable collection of bounded intervals $\{I^k\}_{k=1}^{\infty}$. We first consider the case that I is a closed interval and each I^k is open. By the Heine-Borel Theorem, there is a finite subcover, say $\{I^k\}_{k=1}^{m}$, of I (with m large enough, we can assume the subcover involves the first m intervals).

Proposition 20.10 (continued 1)

Proof (continued). Dividing both sides of this by ε^n and letting $\varepsilon \to 0$ we get, by Lemma 20.8,

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Proposition 20.10 (continued 2)

Proof (continued). Since $\mu^{integral}$ is "clearly" finitely additive and monotone, then

$$\mu^{integral}(I) \leq \mu^{integral}\left(\cup_{k=1}^{\infty} I^k\right) \leq \sum_{k=1}^{m} \mu^{integral}(I^k)$$

where the first inequality holds by monotonicity and the second inequality ("finite monotonicity") follows by decomposing $\cup_{k=1}^{m} I^k$ into disjoint pieces which are subsets of the I^k's (which can be done since the intervals form a semiring) and using finite additivity and monotonicity; details are to be given in Exercise 20.2.A. So by dilating the intervals we get

$$\mu^{integral}(T_\varepsilon(I)) \leq \sum_{k=1}^{m} \mu^{integral}(T_\varepsilon(I^k)) \text{ for all } \varepsilon > 0.$$

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$$\mu^{integral}(T_\varepsilon(I)) \leq \sum_{k=1}^{m} \mu^{integral}(T_\varepsilon(I^k)) \text{ for all } \varepsilon > 0.$$

Proposition 20.10 (continued 3)

Proof (continued). Dividing each side of this inequality by ε^n and letting $\varepsilon \to \infty$ we get from Lemma 20.8 that

$$\text{vol}(I) \leq \sum_{k=1}^{m} \text{vol}(I^k) \leq \sum_{k=1}^{\infty} \text{vol}(I^k),$$

so countable monotonicity holds in this special case.

Now for the general $\{I^k\}_{k=1}^{\infty}$ of bounded intervals in $\mathbb{R}^n$ (not necessarily open) that cover interval I, let $\varepsilon > 0$. Choose a closed interval $(\hat{I})$ that is contained in I with $\text{vol}(I) = \text{vol}(\hat{I}) < \varepsilon$ (just shorten the n intervals from $\mathbb{R}$ that constitute I by a length of $\varepsilon/(n+1)$ each and include the endpoints).

Proposition 20.10 (continued 3)

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Proposition 20.10 (continued 3)

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Proposition 20.10 (continued 4)

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Proof (continued). Therefore

$$\text{vol}(I) < \text{vol}(\hat{I}) + \varepsilon \leq \sum_{k=1}^{\infty} \text{vol}(\hat{I}^k) + \varepsilon$$

$$< \sum_{k=1}^{\infty} \left(\text{vol}(I^k) + \frac{\varepsilon}{2^k} \right) + \varepsilon = \sum_{k=1}^{\infty} \text{vol}(I^k) + 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, then this implies that $\text{vol}(I) \leq \sum_{k=1}^{\infty} \text{vol}(I^k)$. So countable monotonicity holds in general and hence vol is a premeasure on the semiring of intervals in $\mathbb{R}^n$ □

Theorem 20.11

Theorem 20.11. The σ-algebra $\mathcal{L}^n$ of Lebesgue measurable subsets of $\mathbb{R}^n$ contains the bounded intervals in $\mathbb{R}^n$ and contains the Borel subsets in $\mathbb{R}^n$. Moreover, the measure space $(\mathbb{R}^n, \mathcal{L}^n, \mu_n)$ is both σ-finite and complete. For bounded interval I in $\mathbb{R}^n$, $\mu_n(I) = \text{vol}(I)$.

Proof. By Proposition 2.10, volume ("vol") is a premeasure on the semiring of bounded intervals in $\mathbb{R}^n$. Recall that a measure is σ-finite if the whole space is the union of a countable collection of measureable sets, each of finite measure. Here, $\mathbb{R}^n$ can be written as a countable union of intervals (say a countable collection of "cubes" of volume 1), so measure vol is σ-finite.

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Proof (continued). Finally, we show that each Borel set is Lebesgue measurable. Since $\mathcal{L}^n$ is a σ-algebra and the Borel sets form the smallest σ-algebra containing the open sets, it suffices to show that every open set $\mathcal{O}$ of $\mathbb{R}^n$ is Lebesgue measurable. The collection of points in such $\mathcal{O}$ that have rational coordinates is a countable dense subset of $\mathcal{O}$. Let $\{z_k\}_{k=1}^{\infty}$ be an enumeration of this collection. For each k, consider the open cube $I_{k,n}$ centered at q_k of edge length $1/n$ (a "cube" is a Cartesian product of n intervals in $\mathbb{R}$ of the same length). In Exercise 20.16 it is to be shown that $\mathcal{O} = \cup_{I_{k,n} \subset \mathcal{O}} I_{k,n}$. Since each $I_{k,n}$ is an interval in $\mathbb{R}^n$ then each is measurable and since $\mathcal{L}^n$ is a σ-algebra then $\mathcal{O}$ is measurable. Therefore $\mathcal{L}^n$ contains all Borel sets, as claimed. □

Theorem 20.11 (continued)

Theorem 20.11. The σ-algebra $\mathcal{L}^n$ of Lebesgue measurable subsets of $\mathbb{R}^n$ contains the bounded intervals in $\mathbb{R}^n$ and contains the Borel subsets in $\mathbb{R}^n$. Moreover, the measure space $(\mathbb{R}^n, \mathcal{L}^n, \mu_n)$ is both σ-finite and complete. For bounded interval I in $\mathbb{R}^n$, $\mu_n(I) = \text{vol}(I)$.

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Corollary 20.12

Corollary 20.12. Let E be a Lebesgue measurable subset of $\mathbb{R}^n$ and $f : E \to \mathbb{R}$ be continuous. Then f is measurable with respect to n-dimensional Lebesgue measure.

Proof. Let $\mathcal{O}$ be an open set of real numbers. Since f is continuous on E then $f^{-1}(\mathcal{O})$ is open relative to E, say $f^{-1}(\mathcal{O}) = E \cap \mathcal{U}$ where $\mathcal{U}$ is open in $\mathbb{R}^n$. By Theorem 20.11, $\mathcal{U} \subset \mathbb{R}^n$ is measurable (since $\mathcal{L}^n$ includes are Borel, and hence all open, sets). So $f^{-1}(\mathcal{O}) = E \cap \mathcal{U}$ is measurable. By Proposition 18.2, f is measurable, as claimed. □

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Theorem 20.13

Theorem 20.13. Let E be a Lebesgue measurable subset of $\mathbb{R}^n$. Then

$$\mu(E) = \inf\{\mu_n(\mathcal{O}) \mid E \subset \mathcal{O}, \mathcal{O} \text{ is open}\}$$

and

$$\mu(E) = \sup\{\mu_n(\mathcal{K}) \mid \mathcal{K} \subset E, \mathcal{K} \text{ is compact}\}.$$

Proof. We first consider the case in which E is bounded and hence of finite Lebesgue measure. Let $\varepsilon > 0$. Since $\mu_n(E) = \mu_n^*(E) < \infty$, by the definition of Lebesgue outer measure, there is a countable collection of bounded intervals in $\mathbb{R}^n$, $\{I^m\}_{m=1}^\infty$, which covers E and $\sum_{m=1}^\infty \mu_n(I^m) < \mu_n(E) + \varepsilon/2$ by Theorem 0.3, "Epsilon Property of Sup and Inf."

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Theorem 20.13 (continued 1)

Proof (continued). Then $E \subset \mathcal{O}$ and since the measure is countable monotone (by Proposition 17.1)

$$\mu_n(\mathcal{O}) \leq \sum_{m=1}^{\infty} \left(\mu_n(I^k) + \frac{\varepsilon}{2^{m+1}} \right)$$

$$= \sum_{m=1}^{\infty} \mu_n(I^m) + \frac{\varepsilon}{2} < \left(\mu_n(E) + \frac{\varepsilon}{2} \right) + \frac{\varepsilon}{2} = \mu_n(E) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, then $\inf\{\mu_n(\mathcal{O}) \mid E \subset \mathcal{O}, \mathcal{O} \text{ open}\} \leq \mu_n(E)$. By monotonicity (given by Proposition 17.1), for $E \subset \mathcal{O}$ we have $\mu_n(E) \leq \mu_n(\mathcal{O})$, so $\inf\{\mu_n(\mathcal{O}) \mid E \subset \mathcal{O}, \mathcal{O} \text{ open}\} \geq \mu_n(E)$ and we have equality for E bounded and of finite measure.

Theorem 20.13 (continued 1)

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$$\mu_n(\mathcal{O}) \leq \sum_{m=1}^{\infty} \left(\mu_n(I^k) + \frac{\varepsilon}{2^{m+1}} \right)$$

$$= \sum_{m=1}^{\infty} \mu_n(I^m) + \frac{\varepsilon}{2} < \left(\mu_n(E) + \frac{\varepsilon}{2} \right) + \frac{\varepsilon}{2} = \mu_n(E) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, then $\inf\{\mu_n(\mathcal{O}) \mid E \subset \mathcal{O}, \mathcal{O} \text{ open}\} \leq \mu_n(E)$. By monotonicity (given by Proposition 17.1), for $E \subset \mathcal{O}$ we have $\mu_n(E) \leq \mu_n(\mathcal{O})$, so $\inf\{\mu_n(\mathcal{O}) \mid E \subset \mathcal{O}, \mathcal{O} \text{ open}\} \geq \mu_n(E)$ and we have equality for E bounded and of finite measure. Exercise 20.2.B(a) covers the case where E is unbounded and of finite measure, and Exercise 20.2.B(b) covers the case where E is of infinite measure.

Theorem 20.13 (continued 1)

Proof (continued). Then $E \subset \mathcal{O}$ and since the measure is countable monotone (by Proposition 17.1)

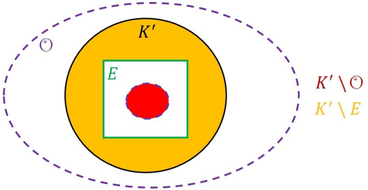
$$\mu_n(\mathcal{O}) \le \sum_{m=1}^{\infty} \left(\mu_n(I^k) + \frac{\varepsilon}{2^{m+1}}\right)$$

$$= \sum_{m=1}^{\infty} \mu_n(I^m) + \frac{\varepsilon}{2} < \left(\mu_n(E) + \frac{\varepsilon}{2}\right) + \frac{\varepsilon}{2} = \mu_n(E) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, then $\inf\{\mu_n(\mathcal{O}) \mid E \subset \mathcal{O}, \mathcal{O} \text{ open}\} \le \mu_n(E)$. By monotonicity (given by Proposition 17.1), for $E \subset \mathcal{O}$ we have $\mu_n(E) \le \mu_n(\mathcal{O})$, so $\inf\{\mu_n(\mathcal{O}) \mid E \subset \mathcal{O}, \mathcal{O} \text{ open}\} \ge \mu_n(E)$ and we have equality for E bounded and of finite measure. Exercise 20.2.B(a) covers the case where E is unbounded and of finite measure, and Exercise 20.2.B(b) covers the case where E is of infinite measure.

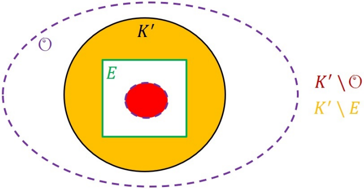
Theorem 20.13 (continued 2)

Proof (continued). For the second claim, we again first consider the case in which E is bounded and hence of finite Lebesgue measure. Since E is bounded, we may choose a closed and bounded set K' that contains E. Since $K' \setminus E$ is bounded, we know from the first claim (and the first part of the proof) that there is an open set $\mathcal{O}$ for which $K' \setminus E \subset \mathcal{O}$ and, by the excision property of μ_n (Prop. 17.1): $\mu_n(\mathcal{O} \setminus (K' \setminus E)) < \varepsilon$. (19)
Define $K = K' \setminus \mathcal{O}$. Then K is closed and bounded in $\mathbb{R}^n$ and therefore compact by the Heine-Borel Theorem. Since $K' \setminus E \subset \mathcal{O}$ and $E \subset K'$ then $K = K' \setminus \mathcal{O} \subset K' \setminus (K' \setminus E) = K' \cap E \subset E$, or $K \subset E$. On the other hand, since $E \subset K'$ we infer that $E \setminus K = E \setminus (K' \setminus \mathcal{O}) = E \cap \mathcal{O}$ and $E \cap \mathcal{O} \subset \mathcal{O} \setminus (K' \setminus E)$.



Theorem 20.13 (continued 2)

Proof (continued). For the second claim, we again first consider the case in which E is bounded and hence of finite Lebesgue measure. Since E is bounded, we may choose a closed and bounded set K' that contains E. Since $K' \setminus E$ is bounded, we know from the first claim (and the first part of the proof) that there is an open set $\mathcal{O}$ for which $K' \setminus E \subset \mathcal{O}$ and, by the excision property of μ_n (Prop. 17.1): $\mu_n(\mathcal{O} \setminus (K' \setminus E)) < \varepsilon$. (19)
Define $K = K' \setminus \mathcal{O}$. Then K is closed and bounded in $\mathbb{R}^n$ and therefore compact by the Heine-Borel Theorem. Since $K' \setminus E \subset \mathcal{O}$ and $E \subset K'$ then $K = K' \setminus \mathcal{O} \subset K' \setminus (K' \setminus E) = K' \cap E \subset E$, or $K \subset E$. On the other hand, since $E \subset K'$ we infer that $E \setminus K = E \setminus (K' \setminus \mathcal{O}) = E \cap \mathcal{O}$ and $E \cap \mathcal{O} \subset \mathcal{O} \setminus (K' \setminus E)$.



Theorem 20.13 (continued 3)

Proof (continued). Therefore, by the excision and monotonicity properties of measure (Proposition 17.1) and (19)

$$\begin{aligned} 0 &\leq \mu_n(E) = \mu_n(K) = \mu_n(E \setminus K) \text{ by excision} \\ &\leq \mu_n(\mathcal{O} \setminus (K' \setminus E)) \text{ since } E \setminus K \subset \mathcal{O} \setminus (K' \setminus E) \\ &< \varepsilon \text{ by (19).} \end{aligned}$$

That is, $\mu_n(E) < \mu_N(K) + \varepsilon$. Since $\varepsilon > 0$ if arbitrary, then $\sup\{\mu_n(\mathcal{K}) \mid \mathcal{K} \subset E, \mathcal{K} \text{ is compact}\} \geq \mu_n(E)$. Since $K \subset E$ implies $\mu_n(K) \leq \mu_n(E)$ by monotonicity (Proposition 17.1) then $\sup\{\mu_n(\mathcal{K}) \mid \mathcal{K} \subset E, \mathcal{K} \text{ is compact}\} \leq \mu_n(E)$ and hence the claim holds for E bounded. Exercise 20.2.C(a) covers the case where E is unbounded and of finite measure, and Exercise 20.2.C(b) covers the case where E is of infinite measure. □

Theorem 20.13 (continued 3)

Proof (continued). Therefore, by the excision and monotonicity properties of measure (Proposition 17.1) and (19)

$$\begin{aligned} 0 &\leq \mu_n(E) = \mu_n(K) = \mu_n(E \setminus K) \text{ by excision} \\ &\leq \mu_n(\mathcal{O} \setminus (K' \setminus E)) \text{ since } E \setminus K \subset \mathcal{O} \setminus (K' \setminus E) \\ &< \varepsilon \text{ by (19).} \end{aligned}$$

That is, $\mu_n(E) < \mu_N(K) + \varepsilon$. Since $\varepsilon > 0$ if arbitrary, then $\sup\{\mu_n(\mathcal{K}) \mid \mathcal{K} \subset E, \mathcal{K} \text{ is compact}\} \geq \mu_n(E)$. Since $K \subset E$ implies $\mu_n(K) \leq \mu_n(E)$ by monotonicity (Proposition 17.1) then $\sup\{\mu_n(\mathcal{K}) \mid \mathcal{K} \subset E, \mathcal{K} \text{ is compact}\} \leq \mu_n(E)$ and hence the claim holds for E bounded. Exercise 20.2.C(a) covers the case where E is unbounded and of finite measure, and Exercise 20.2.C(b) covers the case where E is of infinite measure. □

Theorem 20.15

Proposition 20.15. For the mapping $\varphi : \mathbb{R}^n \to \mathbb{R}^m \times \mathbb{R}^k$ defined by (20), a subset E of $\mathbb{R}^n$ is measurable with respect to n-dimensional Lebesgue measure μ_n if and only if its image $\varphi(E)$ is measurable with respect to the product measure $\mu_m \times \mu_k$ on $\mathbb{R}^m \times \mathbb{R}^k$ and $\mu_n(E) = (\mu_m \times \mu_k)(\varphi(E))$.

Proof. Define $\mathcal{I}_n$ to be the collection of bounded intervals in $\mathbb{R}^n$ and vol_n the set function volume ("vol") defined on $\mathcal{I}_n$. Since vol_n is a σ-finite premeasure by Proposition 20.10 and Theorem 20.11, it follows from the uniqueness part of the Carathédory-Hahn Theorem that Lebesgue measure μ_n is the unique measure on $\mathcal{L}^n$ which extends $\text{vol}_n : \mathcal{I}_n \to [0, \infty]$.

Theorem 20.15

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Proof. Define $\mathcal{I}_n$ to be the collection of bounded intervals in $\mathbb{R}^n$ and vol_n the set function volume ("vol") defined on $\mathcal{I}_n$. Since vol_n is a σ-finite premeasure by Proposition 20.10 and Theorem 20.11, it follows from the uniqueness part of the Carathédory-Hahn Theorem that Lebesgue measure μ_n is the unique measure on $\mathcal{L}^n$ which extends $\text{vol}_n : \mathcal{I}_n \to [0, \infty]$. For interval $I = I_1 \times I_2 \times \cdots \times I_n$ in $\mathbb{R}^n$, $\text{vol}(I) = \ell(I_1)\ell(I_2)\cdots\ell(I_n)$, so

$$(\mu_m \times \mu_n)(\varphi(U)) = (\mu_m \times \mu_n)((I_1 \times I_2 \times \cdots \times I_n), (I_{m+1} \times I_{m+2} \times \cdots \times \mathcal{I}_{m+k}))$$

$$= \mu_m((I_1 \times I_2 \times \cdots \times I_m))\mu_k((I_{m+1} \times I_{m+2} \times \cdots \times I_{m+k}))$$

$$= (\ell(I_1)\ell(I_2)\cdots\ell(I_m))(\ell(I_{m+1})\ell(I_{m+2})\cdots\ell(I_{m+k}))$$

$$= \ell(I_1)\ell(I_2)\cdots\ell(I_n) = \text{vol}_n(I) = \mu_n(I).$$

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Proof. Define $\mathcal{I}_n$ to be the collection of bounded intervals in $\mathbb{R}^n$ and vol_n the set function volume ("vol") defined on $\mathcal{I}_n$. Since vol_n is a σ-finite premeasure by Proposition 20.10 and Theorem 20.11, it follows from the uniqueness part of the Carathédory-Hahn Theorem that Lebesgue measure μ_n is the unique measure on $\mathcal{L}^n$ which extends $\text{vol}_n : \mathcal{I}_n \to [0, \infty]$. For interval $I = I_1 \times I_2 \times \cdots \times I_n$ in $\mathbb{R}^n$, $\text{vol}(I) = \ell(I_1)\ell(I_2)\cdots\ell(I_n)$, so

$$\begin{aligned}(\mu_m \times \mu_n)(\varphi(U)) &= (\mu_m \times \mu_n)((I_1 \times I_2 \times \cdots \times I_n), (I_{m+1} \times I_{m+2} \times \cdots \times \mathcal{I}_{m+k}))\\ &= \mu_m((I_1 \times I_2 \times \cdots \times I_m))\mu_k((I_{m+1} \times I_{m+2} \times \cdots \times I_{m+k}))\\ &= (\ell(I_1)\ell(I_2)\cdots\ell(I_m))(\ell(I_{m+1})\ell(I_{m+2})\cdots\ell(I_{m+k}))\\ &= \ell(I_1)\ell(I_2)\cdots\ell(I_n) = \text{vol}_n(I) = \mu_n(I).\end{aligned}$$

Theorem 20.15 (continued)

Proof (continued). That is, measures of intervals are preserved under φ. It follows that outer measures are also preserved under φ (we leave the details of this claim to Exercise 20.2.D). Since φ is one to one and one, $E \in \mathcal{L}^n$ if and only if $\varphi(E)$ is $(\mu_m \times \mu_k)$-measurable. This establishes the measurability claim, but we still need to establish the equalities of the measures. If we define $\mu'(E) = (\mu_m \times \mu_k)(\varphi(E))$ for all $E \in \mathcal{L}^n$, then μ' is a measure in $\mathcal{L}^n$ and $\mu'(I) = (\mu_m \times \mu_k)(\varphi(I)) = \mu_n(I) = \text{vol}_n(I)$ for all intervals $I \subset \mathbb{R}^n$. So μ' extends vol_n on the collection of bounded intervals in $\mathbb{R}^n$. But, as mentioned above, such an extension is unique (by the Carathédory-Hahn Theorem) so $\mu' = \mu$ on $\mathcal{L}^n$ and

$$\mu_n(E) = \mu'(E) = (\mu_m \times \mu_k)(\varphi(E)) \text{ for all } E \in \mathcal{L}^n.$$

That is, the value of $\mu_n(E)$ equals $(\mu_m \times \mu_k)(\varphi(E))$, as claimed. □

Theorem 20.15 (continued)

Proof (continued). That is, measures of intervals are preserved under φ. It follows that outer measures are also preserved under φ (we leave the details of this claim to Exercise 20.2.D). Since φ is one to one and one, $E \in \mathcal{L}^n$ if and only if $\varphi(E)$ is $(\mu_m \times \mu_k)$-measurable. This establishes the measurability claim, but we still need to establish the equalities of the measures. If we define $\mu'(E) = (\mu_m \times \mu_k)(\varphi(E))$ for all $E \in \mathcal{L}^n$, then μ' is a measure in $\mathcal{L}^n$ and $\mu'(I) = (\mu_m \times \mu_k)(\varphi(I)) = \mu_n(I) = \mathrm{vol}_n(I)$ for all intervals $I \subset \mathbb{R}^n$. So μ' extends vol_n on the collection of bounded intervals in $\mathbb{R}^n$. But, as mentioned above, such an extension is unique (by the Carathédory-Hahn Theorem) so $\mu' = \mu$ on $\mathcal{L}^n$ and

$$\mu_n(E) = \mu'(E) = (\mu_m \times \mu_k)(\varphi(E)) \text{ for all } E \in \mathcal{L}^n.$$

That is, the value of $\mu_n(E)$ equals $(\mu_m \times \mu_k)(\varphi(E))$, as claimed. □

Theorem 20.17

Proposition 20.17. A linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz.

Proof. Let $x \in \mathbb{R}^n$, say $x = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n$ where $e_1, e_2, \ldots, e_n$ are the standard basis vectors in $\mathbb{R}^n$. Then

$$T(x) = T(x_1 e_1 + x_2 e_2 + \cdots + x_n e_n) = x_1 T(e_1) + x_2 T(e_2) + \cdots + x_n T(e_n).$$

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By the Triangle Inequality and positive homogeneity of the norm (see the definition of norm in Section 7.1),

$$\|T(x)\| = \|x_1 T(e_1) + x_2 T(e_2) + \cdots + x_n T(e_n)\|$$

$$\leq \|x_1 T(e_1)\| + \|x_2 T(e_2)\| + \cdots + \|x_n T(e_n)\|$$

$$= |x_1|\|T(e_1)\| + |x_2|\|T(e_2)\| + \cdots + |x_n|\|T(e_n)\| = \sum_{k=1}^{n} |x_k|\|T(e_k)\|.$$

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Theorem 20.17 (continued)

Proof (continued). So with $c = \sqrt{\sum_{k=1}^{n} \|T(e_k)\|^2}$ we have for every $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ by the Cauchy-Schwarz Inequality (which in $\mathbb{R}^n$ states $|x \cdot y| \leq \|x\|\|y\|$ for all $x, y \in \mathbb{R}^n$) that

$$\begin{aligned}
\|T(x)\| &\leq \sum_{k=1}^{n} |x_k|\|T(e_k)\| \\
&= (|x_1|, |x_2|, \ldots, |x_n|) \cdot (\|T(e_1)\|, \|T(e_2)\|, \ldots, \|T(e_n)\|) \\
&\leq \|(|x_1|, |x_2|, \ldots, |x_n|)\|\|(\|T(e_1)\|, \|T(e_2)\|, \ldots, \|T(e_n)\|)\| \\
&= \left\{\sum_{k=1}^{n} |x_k|^2\right\}^{1/2} \left\{\sum_{k=1}^{n} \|T(e_k)\|^2\right\}^{1/2} = c\|x\|.
\end{aligned}$$

So for $u, v \in \mathbb{R}^n$, let $x = u - v$. Then $T(x) = T(u - v) = T(u_- T(v)$ and $\|T(x)\| = \|T(u) - T(v)\| \leq c\|x\| = c\|u - v\|$ and so T is Lipschitz, as claimed. □

Theorem 20.17 (continued)

Proof (continued). So with $c = \sqrt{\sum_{k=1}^{n} \|T(e_k)\|^2}$ we have for every $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ by the Cauchy-Schwarz Inequality (which in $\mathbb{R}^n$ states $|x \cdot y| \leq \|x\|\|y\|$ for all $x, y \in \mathbb{R}^n$) that

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&= (|x_1|, |x_2|, \ldots, |x_n|) \cdot (\|T(e_1)\|, \|T(e_2)\|, \ldots, \|T(e_n)\|) \\
&\leq \|(|x_1|, |x_2|, \ldots, |x_n|)\| \|(\|T(e_1)\|, \|T(e_2)\|, \ldots, \|T(e_n)\|)\| \\
&= \left\{ \sum_{k=1}^{n} |x_k|^2 \right\}^{1/2} \left\{ \sum_{k=1}^{n} \|T(e_k)\|^2 \right\}^{1/2} = c\|x\|.
\end{aligned}$$

So for $u, v \in \mathbb{R}^n$, let $x = u - v$. Then $T(x) = T(u - v) = T(u_- T(v)$ and $\|T(x)\| = \|T(u) - T(v)\| \leq c\|x\| = c\|u - v\|$ and so T is Lipschitz, as claimed. □

Proposition 20.18

Proposition 20.18. Let the mapping $\Psi : \mathbb{R}^n \to \mathbb{R}^n$ be Lipschitz. If E is a Lebesgue measurable subset in $\mathbb{R}^n$, so is $\Psi(E)$. In particular, a linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$ maps Lebesgue measurable sets to Lebesgue measurable sets.

Proof. By the Heine-Borel Theorem, a subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded. A continuous function maps compact sets to compact sets by Proposition 11.20. Since ψ is Lipschitz then, as commented above, it is continuous. For any function f, we have $f(\cup A_i) = \cup f(A_i)$. So ψ maps bounded F_σ sets to F_σ sets.

Proposition 20.18

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Let E be a Lebesgue measurable subset of $\mathbb{R}^n$. Since $\mathbb{R}^n$ is the inion of a countable collection of bounded measurable sets (say unit cubes which tile $\mathbb{R}^n$), we may assume without loss of generality that E is bounded (we take the intersection of E with each unit cube then apply the above mentioned result that $f(\cup A_i) = \cup f(A_i)$). By Corollary 20.14, $E = A \cup D$ where A is an F_σ set and D has Lebesgue out measure zero.

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Proof. By the Heine-Borel Theorem, a subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded. A continuous function maps compact sets to compact sets by Proposition 11.20. Since ψ is Lipschitz then, as commented above, it is continuous. For any function f, we have $f(\cup A_i) = \cup f(A_i)$. So ψ maps bounded F_σ sets to F_σ sets.

Let E be a Lebesgue measurable subset of $\mathbb{R}^n$. Since $\mathbb{R}^n$ is the inion of a countable collection of bounded measurable sets (say unit cubes which tile $\mathbb{R}^n$), we may assume without loss of generality that E is bounded (we take the intersection of E with each unit cube then apply the above mentioned result that $f(\cup A_i) = \cup f(A_i)$). By Corollary 20.14, $E = A \cup D$ where A is an F_σ set and D has Lebesgue out measure zero.

Proposition 20.18 (continued)

Proof (continued). We know from the first paragraph of the proof that $\psi(E)$ is Lebesgue measurable it suffices to show that the set $\psi(D)$ is measurable it suffices to show that the set $\psi(D)$ is measurable. We do so by showing that $\psi(D)$ has Lebesgue outer measure zero.

Let $c > 0$ be such that $\|\psi(u) - \psi(v)\| \leq c\|u - v\|$ for all $u, v \in \mathbb{R}^n$. By Exercise 20.24, there is constant c' that depends only on c and n such that for any interval I in $\mathbb{R}^n$ we have $\mu_n^*(\psi(I)) \leq c'\text{vol}(I)$. Let $\varepsilon > 0$. Since the outer measure $\mu_n^*(D) = 0$, there is a countable collection $\{I^k\}_{k=1}^\infty$, of intervals in $\mathbb{R}^n$ that cover D and for which $\sum_{k=1}^\infty \text{vol}(I^k) < \varepsilon/c'$. Then $\{\psi(I^k)\|_{k=1}^\infty$ is a countable cover of $\psi(D)$.

Proposition 20.18 (continued)

Proof (continued). We know from the first paragraph of the proof that $\psi(E)$ is Lebesgue measurable it suffices to show that the set $\psi(D)$ is measurable it suffices to show that the set $\psi(D)$ is measurable. We do so by showing that $\psi(D)$ has Lebesgue outer measure zero.

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$$\psi_n^*(\psi(D)) < \sum_{k=1}^\infty \mu_n^*(\psi(I^k)) \leq \sum_{k=1}^n c'\text{vol}(I^k) = c' \sum_{k=1}^n \text{vol}(I^k) = c'(\varepsilon/c') = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, then $\mu_n^*(\psi(D)) = 0$ and hence $\psi(D)$ is μ_n-measurable and hence ψ maps Lebesgue measurable sets to Lebesgue measurable sets, as claimed. □

Proposition 20.18 (continued)

Proof (continued). We know from the first paragraph of the proof that $\psi(E)$ is Lebesgue measurable it suffices to show that the set $\psi(D)$ is measurable it suffices to show that the set $\psi(D)$ is measurable. We do so by showing that $\psi(D)$ has Lebesgue outer measure zero.

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$$\psi_n^*(\psi(D)) < \sum_{k=1}^{\infty} \mu_n^*(\psi(I^k)) \leq \sum_{k=1}^{n} c'\text{vol}(I^k) = c' \sum_{k=1}^{n} \text{vol}(I^k) = c'(\varepsilon/c') = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, then $\mu_n^*(\psi(D)) = 0$ and hence $\psi(D)$ is μ_n-measurable and hence ψ maps Lebesgue measurable sets to Lebesgue measurable sets, as claimed. □

Corollary 20.19

Corollary 20.19. Let the function $f : \mathbb{R}^n \to \mathbb{R}$ be measurable with respect to Lebesgue measure and let the operator $T : \mathbb{R}^n \to \mathbb{R}^n$ be linear and invertible. Then the composition $f \circ T : \mathbb{R}^n \to \mathbb{R}$ is also measurable with respect to Lebesgue measure.

Proof. By Proposition 18.2, it is sufficient to show that $(f \circ T)^{-1}(\mathcal{O})$ is Lebesgue measurable for every open $\mathcal{O} \subset \mathbb{R}$. Let $\mathcal{O} \subset \mathbb{R}$ by open. Since $f : \mathbb{R}^n \to \mathbb{R}$ is measurable then by Proposition 18.2 $f^{-1}(\mathcal{O}) \subset \mathbb{R}^n$ is open. Since T^{-1} is linear (because T is) then by Proposition 20.18, $T^{-1}(f^{-1}(\mathcal{O})) \subset \mathbb{R}^n$ is measurable. That is, $(f \circ T)^{-1}(\mathcal{O}) = (T^{-1} \circ f^{-1})(\mathcal{O} = T^{-1}(f^{-1}(\mathcal{O}))$ is measurable and hence $f \circ T$ is a measurable function by Proposition 18.2. □

Corollary 20.19

Corollary 20.19. Let the function $f : \mathbb{R}^n \to \mathbb{R}$ be measurable with respect to Lebesgue measure and let the operator $T : \mathbb{R}^n \to \mathbb{R}^n$ be linear and invertible. Then the composition $f \circ T : \mathbb{R}^n \to \mathbb{R}$ is also measurable with respect to Lebesgue measure.

Proof. By Proposition 18.2, it is sufficient to show that $(f \circ T)^{-1}(\mathcal{O})$ is Lebesgue measurable for every open $\mathcal{O} \subset \mathbb{R}$. Let $\mathcal{O} \subset \mathbb{R}$ by open. Since $f : \mathbb{R}^n \to \mathbb{R}$ is measurable then by Proposition 18.2 $f^{-1}(\mathcal{O}) \subset \mathbb{R}^n$ is open. Since T^{-1} is linear (because T is) then by Proposition 20.18, $T^{-1}(f^{-1}(\mathcal{O})) \subset \mathbb{R}^n$ is measurable. That is, $(f \circ T)^{-1}(\mathcal{O}) = (T^{-1} \circ f^{-1})(\mathcal{O} = T^{-1}(f^{-1}(\mathcal{O}))$ is measurable and hence $f \circ T$ is a measurable function by Proposition 18.2. □

Proposition 20.20

Proposition 20.20. Let $f : \mathbb{R} \to \mathbb{R}$ be integrable over $\mathbb{R}$ with respect to one-dimensional Lebesgue measure μ_1. If $\alpha, \beta \in \mathbb{R}$, $\alpha \neq 0$, then

$$\int_{\mathbb{R}} f\, d\mu_1 = |\alpha| \int_{\mathbb{R}} f(\alpha x)\, d\mu_1(x) \text{ and } \int_{\mathbb{R}} f\, d\mu_1 = \int_{\mathbb{R}} f(x+\beta)\, d\mu_1(x).$$

Proof. Since f is integrable over $\mathbb{R}$ then by Proposition 4.14 and the definition of of integrable, both f^- and f^+ are integrable over $\mathbb{R}$. By definition, $\int_{\mathbb{R}} f\, d\mu_1 = \int_{\mathbb{R}} f^+\, d\mu_1 - \int_{\mathbb{R}} f^-\, d\mu_1$, so if we establish the result for nonnegative function f then the result holds in general. So without loss of generality, we may suppose that f is nonnegative. By Exercise 4.3.24(i), there is an increasing sequence $\{\varphi_n\}$ of nonnegative simple functions on $\mathbb{R}$, each of finite support, which converges pointwise on $\mathbb{R}$ to f.

Proposition 20.20

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Proposition 20.20 (continued 1)

Proof (continued). Now for simple function $\varphi = \sum_{i=1}^n a_i \chi_{E_i}$ we have

$$\int_{\mathbb{R}} \varphi(\alpha x)\, d\mu_1(x) = \int_{\mathbb{R}} \left(\sum_{i=1}^n a_i \chi_{E_i}(\alpha x) \right) d\mu_1(x)$$

$$= \sum_{i=1}^n a_i \left(\int_{\mathbb{R}} \chi_{E_i}(\alpha x)\, d\mu_1(x) \right) = \sum_{i=1}^n a_i \frac{1}{|\alpha|} m(E_i)$$

$$= \frac{1}{|\alpha|} \sum_{i=1}^n a_i m(E_i) = \frac{1}{|\alpha|} \int_{\mathbb{R}} \varphi\, d\mu_1$$

and $$\int_{\mathbb{R}} \varphi(x+\beta)\, d\mu_1(x) = \int_{\mathbb{R}} \left(\sum_{i=1}^n a_i \chi_{E_i}(x+\beta) \right) d\mu_1(x)$$

$$= \sum_{i=1}^n a_i \left(\int_{\mathbb{R}} \chi_{E_i}(x+\beta)\, d\mu_1(x) \right) = \sum_{i=1}^n a_i m(E_i) = \int_{\mathbb{R}} \varphi\, d\mu_1$$

so that the result holds for simple functions.

Proposition 20.20 (continued 2)

Proof (continued). By the Monotone Convergence Theorem (of Section 4.3),

$$\int_{\mathbb{R}} f\,d\mu_1 = \lim_{n\to\infty}\left(\int_{\mathbb{R}} \varphi_n\,d\mu_1\right) = \lim_{n\to\infty}\left(|\alpha|\int_{\mathbb{R}} \varphi_n(\alpha x)\,d\mu_1(x)\right)$$

$$= |\alpha|\lim_{n\to\infty}\left(\int_{\mathbb{R}} \varphi_n(\alpha x)\,d\mu_1(x)\right) = |\alpha|\int_{\mathbb{R}} f(\alpha x)\,d\mu_1(x)$$

since $\varphi_n(\alpha x) \to f(\alpha x)$ monotonically and pointwise. Also by the Monotone Convergence Theorem,

$$\int_{\mathbb{R}} f(x+\beta)\,d\mu_1(x) = \lim_{n\to\infty}\left(\int_{\mathbb{R}} \varphi_n(x+\beta)\,d\mu_1(x)\right)$$

$$= \lim_{n\to\infty}\left(\int_{\mathbb{R}} \varphi_n\,d\mu_1\right) = \int_{\mathbb{R}} f\,f\mu_1$$

since $\varphi_n(x+\beta) \to f(x+\beta)$ monotonically and pointwise. So the result holds for f nonnegative and, as commented above, holds for general integrable f, as claimed. □

Proposition 20.21

Proposition 20.21. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be integrable over $\mathbb{R}^2$ with respect to Lebesgue measure μ_2 and let $c \neq 0$ be a real number. Define $\varphi : \mathbb{R}^2 \to \mathbb{R}$, $\psi : \mathbb{R}^2 \to \mathbb{R}$, and $\eta : \mathbb{R}^2 \to \mathbb{R}$ by $\varphi(x, y) = f(y, x)$, $\psi(x, y) = f(x, x + y)$, and $\eta(x, y) = f(cx, y)$ for all $(x, y) \in \mathbb{R}^2$. Then φ, ψ, and η are integrable over $\mathbb{R}^2$ with respect to Lebesgue measure μ_2. Moreover,

$$\int_{\mathbb{R}^2} f\, d\mu_2 = \int_{\mathbb{R}^2} \varphi\, d\mu_2 = \int_{\mathbb{R}^2} \psi\, d\mu_2 \text{ and } \int_{\mathbb{R}^2} f\, d\mu_2 = |c| \int_{\mathbb{R}^2} \eta\, d\mu_2.$$

Proof for $\varphi(x, y) = f(y, x)$. By Corollary 20.19, each of φ, ψ, and η are μ_2-measurable. We use Fubini's Theorem and Tonelli's Theorem as stated in Theorem 20.16. As in the proof of Proposition 20,20, we may assume without loss of generality that f is nonnegative.

Proposition 20.21

Proposition 20.21. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be integrable over $\mathbb{R}^2$ with respect to Lebesgue measure μ_2 and let $c \neq 0$ be a real number. Define $\varphi : \mathbb{R}^2 \to \mathbb{R}$, $\psi : \mathbb{R}^2 \to \mathbb{R}$, and $\eta : \mathbb{R}^2 \to \mathbb{R}$ by $\varphi(x,y) = f(y,x)$, $\psi(x,y) = f(x, x+y)$, and $\eta(x,y) = f(cx, y)$ for all $(x,y) \in \mathbb{R}^2$. Then φ, ψ, and η are integrable over $\mathbb{R}^2$ with respect to Lebesgue measure μ_2. Moreover,

$$\int_{\mathbb{R}^2} f\,d\mu_2 = \int_{\mathbb{R}^2} \varphi\,d\mu_2 = \int_{\mathbb{R}^2} \psi\,d\mu_2 \text{ and } \int_{\mathbb{R}^2} f\,d\mu_2 = |c| \int_{\mathbb{R}^2} \eta\,d\mu_2.$$

Proof for $\varphi(x,y) = f(y,x)$. By Corollary 20.19, each of φ, ψ, and η are μ_2-measurable. We use Fubini's Theorem and Tonelli's Theorem as stated in Theorem 20.16. As in the proof of Proposition 20,20, we may assume without loss of generality that f is nonnegative.

Since f is integrable over $\mathbb{R}^2$, by Fubini's Theorem (as given in Theorem 20.16),

$$\int_{\mathbb{R}^2} f\,\mu_2 = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x,y)\,d\mu_1(x) \right)\,d\mu_1(y).$$

Proposition 20.21

Proposition 20.21. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be integrable over $\mathbb{R}^2$ with respect to Lebesgue measure μ_2 and let $c \neq 0$ be a real number. Define $\varphi : \mathbb{R}^2 \to \mathbb{R}$, $\psi : \mathbb{R}^2 \to \mathbb{R}$, and $\eta : \mathbb{R}^2 \to \mathbb{R}$ by $\varphi(x, y) = f(y, x)$, $\psi(x, y) = f(x, x + y)$, and $\eta(x, y) = f(cx, y)$ for all $(x, y) \in \mathbb{R}^2$. Then φ, ψ, and η are integrable over $\mathbb{R}^2$ with respect to Lebesgue measure μ_2. Moreover,

$$\int_{\mathbb{R}^2} f\, d\mu_2 = \int_{\mathbb{R}^2} \varphi\, d\mu_2 = \int_{\mathbb{R}^2} \psi\, d\mu_2 \text{ and } \int_{\mathbb{R}^2} f\, d\mu_2 = |c| \int_{\mathbb{R}^2} \eta\, d\mu_2.$$

Proof for $\varphi(x, y) = f(y, x)$. By Corollary 20.19, each of φ, ψ, and η are μ_2-measurable. We use Fubini's Theorem and Tonelli's Theorem as stated in Theorem 20.16. As in the proof of Proposition 20,20, we may assume without loss of generality that f is nonnegative.

Since f is integrable over $\mathbb{R}^2$, by Fubini's Theorem (as given in Theorem 20.16),

$$\int_{\mathbb{R}^2} f\, \mu_2 = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x, y)\, d\mu_1(x) \right)\, d\mu_1(y).$$

Proposition 20.21 (continued)

Proof (continued). Now $f(x,y) = \varphi)(y,x)$ so $\int_{\mathbb{R}} f(x,y)\,d\mu_1(x) = \int_{\mathbb{R}} \varphi(y,x)\,d\mu_1(x)$ and therefore

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x,y)\,d\mu_1(x) \right) d\mu_1(y) = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \varphi(y,x)\,d\mu_1(x) \right) \, d\mu_1(y).$$

Since φ is nonnegative (because f is nonnegative without loss of generality) and μ_2-measurable, then by Tonelli's Theorem (as given in Theorem 20.16, the "moreover" part)

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \varphi(y,x)\,d\mu_1(x) \right) \, d\mu_1(y) = \int_{\mathbb{R}^2} \varphi\,d\mu_2.$$

Therefore, $\int_{\mathbb{R}^2} f\,d\mu_2 = \int_{\mathbb{R}^2} \varphi\,d\mu_2$, as claimed. □

Proposition 20.22

Proposition 20.22. Let the linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$ be invertible and the function $f : \mathbb{R}^n \to \mathbb{R}$ be integrable over $\mathbb{R}^n$ with respect to Lebesgue measure μ_n. Then the composition $f \circ T : \mathbb{R}^n \to \mathbb{R}$ is also integrable over $\mathbb{R}^n$ with respect to Lebesgue measure μ_n and

$$\int_{\mathbb{R}^n} f \, d\mu_n = |\det(T)| \int_{\mathbb{R}^n} f \circ T \, d\mu_n \text{ or } \int_{\mathbb{R}^n} f \circ T \, d\mu_n = \frac{1}{|\det(T)|} \int_{\mathbb{R}^n} f \, d\mu_n.$$

Proof. As in the proof of Proposition 20.20, we may assume without loss of generality that f is nonnegative. By the multiplicative property of the determinant (Property (i)) and the fact that every invertible linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$ is a composition of linear operators of Types 1, 2, 3 above we need only establish the result for Type 1, 2, 3 linear operators. The cases $n = 1$ and $n = 2$ follow from Propositions 20.20 and 20.21. We now give an inductive proof. Suppose the result has been established for m where $m \geq 2$ and consider the case $n = m + 1$.

Proposition 20.22

Proposition 20.22. Let the linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$ be invertible and the function $f : \mathbb{R}^n \to \mathbb{R}$ be integrable over $\mathbb{R}^n$ with respect to Lebesgue measure μ_n. Then the composition $f \circ T : \mathbb{R}^n \to \mathbb{R}$ is also integrable over $\mathbb{R}^n$ with respect to Lebesgue measure μ_n and

$$\int_{\mathbb{R}^n} f\, d\mu_n = |\det(T)| \int_{\mathbb{R}^n} f \circ T\, d\mu_n \text{ or } \int_{\mathbb{R}^n} f \circ T\, d\mu_n = \frac{1}{|\det(T)|} \int_{\mathbb{R}^n} f\, d\mu_n.$$

Proof. As in the proof of Proposition 20.20, we may assume without loss of generality that f is nonnegative. By the multiplicative property of the determinant (Property (i)) and the fact that every invertible linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$ is a composition of linear operators of Types 1, 2, 3 above we need only establish the result for Type 1, 2, 3 linear operators. The cases $n = 1$ and $n = 2$ follow from Propositions 20.20 and 20.21. We now give an inductive proof. Suppose the result has been established for m where $m \geq 2$ and consider the case $n = m + 1$.

Proposition 20.22 (continued 1)

Proof (continued). Since T is Type 1, 2, or 3 then either (1) $T(e_n) = e_n$ or (2) $T(e_1) = e_1$ (that is, since $n \geq 3$ and T is Type 1, 2, or 3, we cannot have both $T(e_n) \neq e_n$ and $T(e_1) \neq e_1$; notice that T "differs from the identity" by only one "very elementary" operation). In the first case, T maps subspace $\{x \in \mathbb{R}^n \mid x = (x_1, x_2, \ldots, x_{n-1}, 0)\}$ into itself and in the second case T maps the subspace $\{x \in \mathbb{R}^n \mid x = (0, x_2, x_3, \ldots, x_n)\}$ into itself. We consider the first case, with the second case following similarly.

Let T' be the operator induced on $\mathbb{R}^{n-1}$ by T (that is, the restriction of T to $\mathbb{R}^{n=1} = \{x = (x_1, x_2, \ldots, x_n)\}$). By Property (iii), $|\det(T')| = |\det(T)|$. By Corollary 20.19, $f \circ T$ is μ_n measurable.

Proposition 20.22 (continued 1)

Proof (continued). Since T is Type 1, 2, or 3 then either (1) $T(e_n) = e_n$ or (2) $T(e_1) = e_1$ (that is, since $n \geq 3$ and T is Type 1, 2, or 3, we cannot have both $T(e_n) \neq e_n$ and $T(e_1) \neq e_1$; notice that T "differs from the identity" by only one "very elementary" operation). In the first case, T maps subspace $\{x \in \mathbb{R}^n \mid x = (x_1, x_2, \ldots, x_{n-1}, 0)\}$ into itself and in the second case T maps the subspace $\{x \in \mathbb{R}^n \mid x = (0, x_2, x_3, \ldots, x_n)\}$ into itself. We consider the first case, with the second case following similarly.

Let T' be the operator induced on $\mathbb{R}^{n-1}$ by T (that is, the restriction of T to $\mathbb{R}^{n=1} = \{x = (x_1, x_2, \ldots, x_n)\}$). By Property (iii), $|\det(T')| = |\det(T)|$. By Corollary 20.19, $f \circ T$ is μ_n measurable. Since f is integrable and (WLOG) nonnegative, Theorem 20.16 applies and

$$\int_{\mathbb{R}^n} f \circ T \, d\mu_n = \int_{\mathbb{R}} \left(\int_{\mathbb{R}^{n-1}} f \circ T(x_1, x_2, \ldots, x_n) \, d\mu_{n-1}(x_1, x_2, \ldots, x_{n-1}) \right) d\mu_1(x_n)$$

by Tonelli's Theorem (as given in Theorem 20.16) since f and $f \circ T$ are nonnegative

Proposition 20.22 (continued 1)

Proof (continued). Since T is Type 1, 2, or 3 then either (1) $T(e_n) = e_n$ or (2) $T(e_1) = e_1$ (that is, since $n \geq 3$ and T is Type 1, 2, or 3, we cannot have both $T(e_n) \neq e_n$ and $T(e_1) \neq e_1$; notice that T "differs from the identity" by only one "very elementary" operation). In the first case, T maps subspace $\{x \in \mathbb{R}^n \mid x = (x_1, x_2, \ldots, x_{n-1}, 0)\}$ into itself and in the second case T maps the subspace $\{x \in \mathbb{R}^n \mid x = (0, x_2, x_3, \ldots, x_n)\}$ into itself. We consider the first case, with the second case following similarly.

Let T' be the operator induced on $\mathbb{R}^{n-1}$ by T (that is, the restriction of T to $\mathbb{R}^{n=1} = \{x = (x_1, x_2, \ldots, x_n)\}$). By Property (iii), $|\det(T')| = |\det(T)|$. By Corollary 20.19, $f \circ T$ is μ_n measurable. Since f is integrable and (WLOG) nonnegative, Theorem 20.16 applies and

$$\int_{\mathbb{R}^n} f \circ T \, d\mu_n = \int_{\mathbb{R}} \left(\int_{\mathbb{R}^{n-1}} f \circ T(x_1, x_2, \ldots, x_n) \, d\mu_{n-1}(x_1, x_2, \ldots, x_{n-1}) \right)$$

$d\mu_1(x_n)$ by Tonelli's Theorem (as given in Theorem 20.16) since f and $f \circ T$ are nonnegative

Proposition 20.22 (continued 2)

Proof (continued).

$$= \int_{\mathbb{R}} \left(\int_{\mathbb{R}^{n-1}} f(T'((x_1, x_2, \ldots, x_{n-1}), x_n) d\mu_{n-1}(x_1, x_2, \ldots, x_{n-1}) \right)$$

$d\mu_1(x_n)$ since $T' = T$ restricted and $T(e_n) = e_n$

$$= \frac{1}{|\det(T')|} \int_{\mathbb{R}} \left(\int_{\mathbb{R}^{n-1}} f(x_1, x_2, \ldots, x_n)\, d\mu_{n-1}(x_1, x_2, \ldots, x_{n-1}) \right) d\mu_1(x_n)$$

by the induction hypothesis

$$= \frac{1}{|\det(T)|} \int_{\mathbb{R}^n} f\, d\mu_n$$ since $\det(T') = \det(T)$ and by Fubini's Theorem (as given in Theorem 20.16) since f is integrable.

□

Corollary 20.23

Corollary 20.23. Let the linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$ be invertible. Then for each Lebesgue measurable subset E of $\mathbb{R}^n$, $T(E)$ is Lebesgue measurable and $\mu_n(T(E)) = |\det(T)|\mu_n(E)$.

Proof. Let E be bounded. By Proposition 20.17, T is Lipschitz. So $T(E)$ is bounded. By Proposition 20.18, $T(E)$ is Lebesgue measurable and of finite measure since it is bounded. So $f = \chi_{T(E)}$ is integrable over $\mathbb{R}^n$ with respect to Lebesgue measure μ_n.

Corollary 20.23

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Proof. Let E be bounded. By Proposition 20.17, T is Lipschitz. So $T(E)$ is bounded. By Proposition 20.18, $T(E)$ is Lebesgue measurable and of finite measure since it is bounded. So $f = \chi_{T(E)}$ is integrable over $\mathbb{R}^n$ with respect to Lebesgue measure μ_n. Now for $x \in E$, $(f \circ T)(x) = f(T(x)) = \chi_{T(E)}(T(x)) = 1$, and for $x \notin E$, $T(x) \notin E$ (since T is one to on) and $(f \circ T)(x) = f(T(x)) = \chi_{T(E)}(T(x)) = 0$; that is, $f \circ T = \chi_E$. Therefore $\int_{\mathbb{R}^n} f \circ T \, d\mu_n = \int_{\mathbb{R}^n} \chi_E \, d\mu_n = \mu_n(E)$ and $\int_{\mathbb{R}^n} f \, f\mu_n = \int_{\mathbb{R}^n} \chi_{T(E)} \, d\mu_n = \mu_n(T(E))$.

Corollary 20.23

Corollary 20.23. Let the linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$ be invertible. Then for each Lebesgue measurable subset E of $\mathbb{R}^n$, $T(E)$ is Lebesgue measurable and $\mu_n(T(E)) = |\det(T)|\mu_n(E)$.

Proof. Let E be bounded. By Proposition 20.17, T is Lipschitz. So $T(E)$ is bounded. By Proposition 20.18, $T(E)$ is Lebesgue measurable and of finite measure since it is bounded. So $f = \chi_{T(E)}$ is integrable over $\mathbb{R}^n$ with respect to Lebesgue measure μ_n. Now for $x \in E$, $(f \circ T)(x) = f(T(x)) = \chi_{T(E)}(T(x)) = 1$, and for $x \notin E$, $T(x) \notin E$ (since T is one to on) and $(f \circ T)(x) = f(T(x)) = \chi_{T(E)}(T(x)) = 0$; that is, $f \circ T = \chi_E$. Therefore $\int_{\mathbb{R}^n} f \circ T \, d\mu_n = \int_{\mathbb{R}^n} \chi_E \, d\mu_n = \mu_n(E)$ and $\int_{\mathbb{R}^n} f \, f\mu_n = \int_{\mathbb{R}^n} \chi_{T(E)} \, d\mu_n = \mu_n(T(E))$. So by Theorem 20.22,

$$\mu_n(E) = \int_{\mathbb{R}^n} f \circ T \, d\mu_n = \frac{1}{|\det(T)|} \int_{\mathbb{R}^n} f \, d\mu_n = \frac{1}{|\det(T)|} \mu_n(T(E))$$

and the claim holds for E bounded. We leave the case of E unbounded to Exercise 20.2.G. □

Corollary 20.23

Corollary 20.23. Let the linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$ be invertible. Then for each Lebesgue measurable subset E of $\mathbb{R}^n$, $T(E)$ is Lebesgue measurable and $\mu_n(T(E)) = |\det(T)|\mu_n(E)$.

Proof. Let E be bounded. By Proposition 20.17, T is Lipschitz. So $T(E)$ is bounded. By Proposition 20.18, $T(E)$ is Lebesgue measurable and of finite measure since it is bounded. So $f = \chi_{T(E)}$ is integrable over $\mathbb{R}^n$ with respect to Lebesgue measure μ_n. Now for $x \in E$, $(f \circ T)(x) = f(T(x)) = \chi_{T(E)}(T(x)) = 1$, and for $x \notin E$, $T(x) \notin E$ (since T is one to on) and $(f \circ T)(x) = f(T(x)) = \chi_{T(E)}(T(x)) = 0$; that is, $f \circ T = \chi_E$. Therefore $\int_{\mathbb{R}^n} f \circ T \, d\mu_n = \int_{\mathbb{R}^n} \chi_E \, d\mu_n = \mu_n(E)$ and $\int_{\mathbb{R}^n} f \, f\mu_n = \int_{\mathbb{R}^n} \chi_{T(E)} \, d\mu_n = \mu_n(T(E))$. So by Theorem 20.22,

$$\mu_n(E) = \int_{\mathbb{R}^n} f \circ T \, d\mu_n = \frac{1}{|\det(T)|} \int_{\mathbb{R}^n} f \, d\mu_n = \frac{1}{|\det(T)|} \mu_n(T(E))$$

and the claim holds for E bounded. We leave the case of E unbounded to Exercise 20.2.G. □