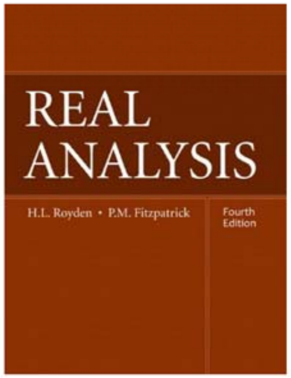
Real Analysis

Chapter 20. The Construction of Particular Measures

20.3. Cumulative Distribution Functions and Borel Measures on $\mathbb{R}$—Proofs of Theorems



Proposition 20.25

Proposition 20.25. Let μ be a Borel measure on $\mathcal{B}(I)$. Then its cumulative distribution function g_μ is increasing and continuous on the right. Conversely, each function $g : I \to \mathbb{R}$ is increasing and continuous on the right is the cumulative distribution function of a unique Borel measure μ_g on $\mathcal{B}(I)$.

Proof. First, let μ be a Borel measure on $\mathcal{B}(I)$. For $x > y$, $g_\mu(x) = \mu([a, x]) \geq \mu([a, y]) = g_\mu(y)$ by monotonicity and so g_μ is increasing. $g_m u$ is also bounded by $g_\mu(b) = \mu([a, b]) = \mu(I)$. Let $x_0 \in [a, b)$ and let $\{x_k\}$ be a decreasing sequence in $(x_0, b]$ that converges to x_0. Then $\cap_{k=1}^\infty (x_0, x_k] = \varnothing$. Since μ is finite, by the Continuity of Measure (Proposition 17.2)

$$\begin{aligned} 0 &= \mu(\varnothing) = \lim_{k\to\infty} \mu((x_0, x_k]) \\ &= \lim_{k\to\infty} (\mu([a, x_k] - g([a, x_0])) \text{ by additivity} \\ &= \lim_{k\to\infty} (g_\mu(x_k) - g_\mu(x_0)). \end{aligned}$$

Proposition 20.25 (continued 1)

Proof (continued). Since $\{x_k\}$ is an arbitrary sequence in $(x_0, b]$ that converges to x_0, then $\lim_{x\to x_0} g_\mu(x) = g_\mu(x_0)$ and g_μ is continuous on the right at x_0.

For the converse, let $g : I \to \mathbb{R}$ be an increasing function that is continuous on the right. Consider the collection $\mathcal{S}$ of subsets of I consisting of the empty set, the singleton set $\{a\}$, and all subintervals of $I = [a, b]$ of the form $(c, d]$. Then $\mathcal{S}$ is a semiring (the intersection of two elements of $\mathcal{S}$ is either $\varnothing$ or an interval of the form $(c, d]$, and the set difference of two elements of $\mathcal{S}$ is either $\varnothing$, and element of the form $(c, d]$, or a set of the form $(c_1, d_1] \cup (c_2, d_2]$). Consider the set function $\mu : \mathcal{S} \to \mathbb{R}$ defined by setting $\mu(\varnothing) = 0$, $\mu(\{a\}) = g(a)$, and $\mu((c, d]) = g(d) - g(c)$ for $(c, d] \subset I$.

Proposition 20.25 (continued 2)

Proof (continued). In Exercise 20.39 (this is where continuity on the right is needed) it is to be shown that if $(c, d] \subset I$ is the union of finite disjoint collection $\cup_{k=1}^\infty (c_k, d_k]$, then $g(d) - g(c) = \sum_{k=1}^n (g(d_k) - g(c_k))$ and that if $(c, d] \subset I$ is covered by the countable collection $\cup_{k=1}^\infty (c_k, d_k]$ then $g(d) - g(c) \leq \sum_{k=1}^\infty (g(d_k) - g(c_k))$. So g is finitely additive and countably monotone on $\mathcal{S}$. Therefore, by definition (see Section 17.5), μ is a premeasure on $\mathcal{S}$. By the Carathéodory-Hahn Theorem (see Section 17.5) the Carathéodory measure $\overline{\mu}$ induced by μ is an extension of μ. Now the μ^* measurable sets form a σ-algebra (by Theorem 17.8) including $\mathcal{S}$ and so for $(c, d) \subset [a, b]$ we have $(c, d) = \cup_{k=1}^\infty (c, d - 1/k]$, and so every open subinterval of $[a, b]$ is μ^*-measurable and hence every open subset of $[a, b]$ (being a countable union of open intervals) is μ^*-measurable. So the μ^*-measurable sets are a σ-algebra containing all open subsets of I, $\mathcal{B}(I)$.

Proposition 20.25 (continued 3)

Proposition 20.25. Let μ be a Borel measure on $\mathcal{B}(I)$. Then its cumulative distribution function g_μ is increasing and continuous on the right. Conversely, each function $g : I \to \mathbb{R}$ is increasing and continuous on the right is the cumulative distribution function of a unique Borel measure μ_g on $\mathcal{B}(I)$.

Proof (continued). For each $x \in [a, b]$,

$$\begin{aligned}
\overline{\mu}([a,x]) &= \mu([a,x]) = \mu(\{a\}) + \mu((a,x]) \text{ by additivity} \\
&= g(a) + (g(x) - g(a)) \text{ since } \mu((c,d]) = g(d) - g(c) \\
&\quad \text{ for } (c,d] \subset I \\
&= g(x).
\end{aligned}$$

So g is the cumulative distribution functions for the restriction of $\overline{\mu}$ is to σ-algebra $\mathcal{B}(I)$, as claimed. □