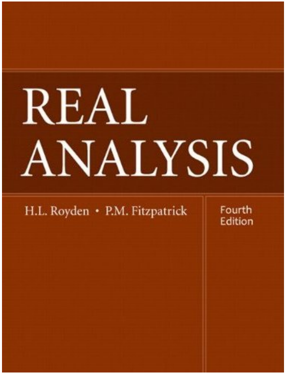
# Real Analysis

**Chapter 20. The Construction of Particular Measures**

20.3. Cumulative Distribution Functions and Borel Measures on $\mathbb{R}$—Proofs of Theorems



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# Proposition 20.25

**Proposition 20.25.** Let $\mu$ be a Borel measure on $\mathcal{B}(I)$. Then its cumulative distribution function $g_\mu$ is increasing and continuous on the right. Conversely, each function $g : I \to \mathbb{R}$ is increasing and continuous on the right is the cumulative distribution function of a unique Borel measure $\mu_g$ on $\mathcal{B}(I)$.

**Proof.** First, let $\mu$ be a Borel measure on $\mathcal{B}(I)$. For $x > y$, $g_\mu(x) = \mu([a, x]) \geq \mu([a, y]) = g_\mu(y)$ by monotonicity and so $g_\mu$ is increasing. $g_m u$ is also bounded by $g_\mu(b) = \mu([a, b]) = \mu(I)$. Let $x_0 \in [a, b)$ and let $\{x_k\}$ be a decreasing sequence in $(x_0, b]$ that converges to $x_0$. Then $\cap_{k=1}^{\infty}(x_0, x_k] = \varnothing$.

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$$\begin{aligned} 0 &= \mu(\varnothing) = \lim_{k\to\infty} \mu((x_0, x_k]) \\ &= \lim_{k\to\infty} (\mu([a,x_k] - g([a,x_0])) \text{ by additivity} \\ &= \lim_{k\to\infty} (g_\mu(x_k) - g_\mu(x_0)). \end{aligned}$$

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# Proposition 20.25 (continued 1)

**Proof (continued).** Since $\{x_k\}$ is an arbitrary sequence in $(x_0, b]$ that converges to $x_0$, then $\lim_{x \to x_0} g_\mu(x) = g_\mu(x_0)$ and $g_\mu$ is continuous on the right at $x_0$.

For the converse, let $g : I \to \mathbb{R}$ be an increasing function that is continuous on the right. Consider the collection $\mathcal{S}$ of subsets of $I$ consisting of the empty set, the singleton set $\{a\}$, and all subintervals of $I = [a, b]$ of the form $(c, d]$. Then $\mathcal{S}$ is a semiring (the intersection of two elements of $\mathcal{S}$ is either $\varnothing$ or an interval of the form $(c, d]$, and the set difference of two elements of $\mathcal{S}$ is either $\varnothing$, and element of the form $(c, d]$, or a set of the form $(c_1, d_1] \cup (c_2, d_2]$). Consider the set function $\mu : \mathcal{S} \to \mathbb{R}$ defined by setting $\mu(\varnothing) = 0$, $\mu(\{a\}) = g(a)$, and $\mu((c, d]) = g(d) - g(c)$ for $(c, d] \subset I$.

# Proposition 20.25 (continued 1)

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# Proposition 20.25 (continued 2)

**Proof (continued).** In Exercise 20.39 (this is where continuity on the right is needed) it is to be shown that if $(c, d] \subset I$ is the union of finite disjoint collection $\cup_{k=1}^{\infty}(c_k, d_k]$, then $g(d) - g(c) = \sum_{k=1}^{n}(g(d_k) - g(c_k))$ and that if $(c, d] \subset I$ is covered by the countable collection $\cup_{k=1}^{\infty}(c_k, d_k]$ then $g(d) - g(c) \leq \sum_{k=1}^{\infty}(g(d_k) - g(c_k))$. So $g$ is finitely additive and countably monotone on $\mathcal{S}$. Therefore, by definition (see Section 17.5), $\mu$ is a premeasure on $\mathcal{S}$. By the Carathéodory-Hahn Theorem (see Section 17.5) the Carathéodory measure $\overline{\mu}$ induced by $\mu$ is an extension of $\mu$. Now the $\mu^*$ measurable sets form a $\sigma$-algebra (by Theorem 17.8) including $\mathcal{S}$ and so for $(c, d) \subset [a, b]$ we have $(c, d) = \cup_{k=1}^{\infty}(c, d - 1/k]$, and so every open subinterval of $[a, b]$ is $\mu^*$-measurable and hence every open subset of $[a, b]$ (being a countable union of open intervals) is $\mu^*$-measurable. So the $\mu^*$-measurable sets are a $\sigma$-algebra containing all open subsets of $I$, $\mathcal{B}(I)$.

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**Proof (continued).** For each $x \in [a, b]$,

$$
\begin{aligned}
\overline{\mu}([a,x]) &= \mu([a,x]) = \mu(\{a\}) + \mu((a,x]) \text{ by additivity} \\
&= g(a) + (g(x) - g(a)) \text{ since } \mu((c,d]) = g(d) - g(c) \\
&\quad\ \text{for } (c,d] \subset I \\
&= g(x).
\end{aligned}
$$

So $g$ is the cumulative distribution functions for the restriction of $\overline{\mu}$ is to $\sigma$-algebra $\mathcal{B}(I)$, as claimed. □