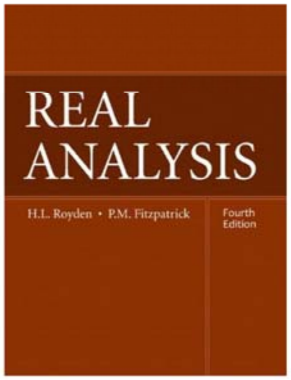
Real Analysis

Chapter 20. The Construction of Particular Measures

20.4. Carathéodory Outer Measure and Hausdorff Measures on a Metric Space—Proofs of Theorems



Proposition 20.27

Proposition 20.27. Let φ be a real-valued function on a set X and $\mu^* : 2^X \to [0, \infty]$ an outer measure with the property that whenever two subsets A and B of X are separated by φ, then

$$\mu^*(A \cup B) = \mu^*(A) + \mu^*(B).$$

Then φ is measurable with respect to the measure induced by μ^*.

Proof. Let $a \in \mathbb{R}$. We show that $E = \{x \in X \mid \varphi(x) > a\}$ is μ^*-measurable, implying the measurability of function φ. By definition, an outer measure is countably monotone (see Section 17.3) so

$$\mu^*(A) = \mu^*((A \cap E) \cup (A \cap E^c)) \leq \mu^*(A \cap E) + \mu^*(A \cap E^c).$$

So we need only show that for A a set of finite measure and for any $\varepsilon > 0$,

$$\mu^*(A) + \varepsilon > \mu^*(A \cap E) + \mu^*(A \cap E^c). \qquad (32)$$

Notice that (32) trivially holds if $\mu^*(A) = \infty$, so we can assume $\mu^*(A) < \infty$. Define $B = A \cap E$ and $C = A \cap E^c$.

Proposition 20.27 (continued 1)

Proof (continued). For each $n \in \mathbb{N}$, define

$$B_n = \{x \in \mid \varphi(x) > a + 1/n\} \text{ and } R_n = B_n \setminus B_{n-1}.$$

Notice that $B_1 \subset B_2 \subset \cdots$, $\cup_{k=1}^{\infty} B_n = B$, and

$$B = b_n \cup \left(\cup_{k=n+1}^{\infty}(B_k \setminus B_{k-1})\right) = B_n \cup \left(\cup_{k=n+1}^{\infty} R_k\right).$$

Now on B_{n-2} (by definition of B_n) we have $\varphi > a + 1/(n-2)$, which on $R_n = B_n \setminus B_{n-1}$ we have $a + 1/n < \varphi \leq a + 1/(n-1)$. Thus φ separates R_n and B_{n-2} and hence separates R_{2k} and $\cup_{j=1}^{k-1} R_{2k}$ since $\cup_{j=1}^{k-1} R_{2j} \subset B_{2k-2}$. So by hypothesis,

$$\mu^*\left(\cup_{j=1}^{k} R_{2j}\right) = \mu^*(R_{2k}) + \mu^*\left(\cup_{j=1}^{k-1} R_{2j}\right).$$

So by induction on k,

$$\mu^*\left(\cup_{j=1}^{k} R_{2j}\right) = \mu^*(R_{2k}) + \mu^*\left(\cup_{j=1}^{k-1} R_{2j}\right) = \sum_{j=1}^{k} \mu^*(R_{2j}).$$

Proposition 20.27 (continued 2)

Proof (continued). Since $\cup_{j=1}^{k} R_{2j} \subset B_{2k} \subset B \subset A$ we have by monotonicity that

$$\mu^*\left(\cup_{j=1}^{k} R_{2j}\right) \leq \mu^*(A) \text{ or } \sum_{j=1}^{k} \mu^*(R_{2j}) \leq \mu^*(A).$$

Since we have $\mu^*(A) < \infty$ then the series $\sum_{j=1}^{\infty} \mu^*(R_{2j})$ converges (absolutely). Similarly, the series $\sum_{j=1}^{\infty} \mu^*(R_k)$ converges. So there is $n \in \mathbb{N}$ such that $\sum_{k=n+1}^{\infty} \mu^*(R_k) < \varepsilon$. Since $B = B_n \cup \left(\cup_{k=n+1}^{\infty} R_k\right)$ then by the countable monotonicity of μ^*,

$$\mu^*(B) \leq \mu^*(B_n) + \sum_{k=n+1}^{\infty} \mu^*(R_k) < \mu^*(B_n) + \varepsilon$$

or $\mu^*(B_n) > \mu^*(B) - \varepsilon$. Now $C = A \cap E^c$ by definition so $C \subset A$, and $B = A \cap E$ by definition so $B \subset A$. So by monotonicity of μ^*, $\mu^*(A) \geq \mu^*(B_n \cup C)$.

Proposition 20.27 (continued 3)

Proposition 20.27. Let φ be a real-valued function on a set X and $\mu^* : 2^X \to [0, \infty]$ an outer measure with the property that whenever two subsets A and B of X are separated by φ, then

$$\mu^*(A \cup B) = \mu^*(A) + \mu^*(B).$$

Then φ is measurable with respect to the measure induced by μ^*.

Proof (continued). Now $\varphi > a + 1/n$ on B_n and $\varphi \le a$ on E^c (by the definition of E) and so φ separates B_n and C. So by hypothesis, $\mu^*(A) \ge \mu^*(B_n \cup C) = \mu^*(B_n) + \mu^*(C)$. Since $\mu^*(B_n) > \mu^*(B) - \varepsilon$ then $\mu^*(A) > \mu^*(B) + \mu^*(C)$, or $\mu^*(A) + \varepsilon > \mu^*(A \cap E) + \mu^*(A \cap E^c)$, and hence $\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$ for all $A \subset X$ where A is of finite outer measure. That is, $E = \{x \in X \mid \varphi(x) > a\}$ is measurable. Since $a \in \mathbb{R}$ is arbitrary, then function φ is measurable. □

Proposition 20.28

Theorem 20.28. Let μ^* be a Carathéodory outer measure on matrix space (X, ρ). Then every Borel subset of X is measurable with respect to μ^*.

Proof. The collection of Borel sets is the smallest σ-algebra containing the closed sets, and the measurable sets are a σ-algebra. So if we show that each closed set is measurable, then the result follows. For closed set F, define function $f(x) = \rho(F, \{x\})$. In Exercise 20.4.A it is to be shown that f is continuous on X and that $f^{-1}(\{0\}) = F$. So if we show f is measurable then this implies $f^{-1}(\{0\}) = \{x \in X \mid f(x) \ge 0\} \cap \{x \in X \mid f(x) \le 0\}$ is measurable (that is, arbitrary closed set F is measurable) and the result then follows. We use Proposition 20.27 to show that every continuous real valued function on X is measurable. First, if a continuous function separates no sets, then the hypothesis of Proposition 20.27 are vacuously satisfies and that function is measurable.

Proposition 20.28 (continued)

Proof. Second, let A and B be subsets of X which are separated by some continuous function f, say $f \le a$ on A and $f \ge b$ on B where $a < b$. ASSUME that $\rho(A, B) = 0$. Then there are sequences $\{u_k\} \subset A$ and $\{v_k\} \subset B$ such that $\lim_{k\to\infty} \rho(u_k, v_k) = 0$. Since f is continuous then $\lim_{k\to\infty} |f(u_k) - f(v_k)| = 0$. But then there is some $u_N \in A$ and $v_N \in B$ with $|f(u_N) - f(v_N)| < (b - a)/2$, a CONTRADICTION. So the assumption that $\rho(A, B) = 0$ is false and in fact $\rho(A, B) > 0$. Since μ^* is a Carathéodory outer measure then (by definition) $\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$. So for *any* $A, B \subset X$ separated by some continuous function f, we have by Proposition 20.27 that continuous f is a measurable function with respect to μ^*. Since this holds for arbitrary A and B, all continuous functions are measurable with respect to μ^*. As discussed above, this establishes that all Borel sets are μ_n^*-measurable. □

Proposition 20.29

Proposition 20.29. Let (X, ρ) be a metric space and α a positive real number. Then $H_\alpha^* : 2^X \to [0, \infty]$ is a Carathéodory outer measure.

Proof. First, for any $\varepsilon > 0$, $H_\alpha^{(\varepsilon)}(\varnothing) = 0$ and so $H_\alpha^*(\varnothing) = 0$. For countable monotonicity, let $\{E_k\}_{k=1}^\infty$ be a countable cover of E. For any coverings $\{A_i^k\}_{i=1}^\infty$ of E_k (for $k = 1, 2, \ldots$) we have $\{A_i^k\}_{i,k=1}^\infty$ is a countable cover of $E \subset \cup_{k=1}^\infty E_k$. Now

$$\sum_{i,k=1}^\infty \left(\operatorname{diam}(A_i^k)\right)^\alpha = \sum_{k=1}^\infty \left(\sum_{i=1}^\infty \left(\operatorname{diam}(A_i^k)\right)^\alpha\right)$$

and taking an infimum over all such A_i^k we get

$$\inf \sum_{i,k=1}^\infty \left(\operatorname{diam}(A_i^k)\right)^\alpha = \inf \left(\sum_{k=1}^\infty \sum_{i=1}^\infty \left(\operatorname{diam}(A_i^k)\right)^\alpha\right)$$

Proposition 20.29 (continued 1)

Proof (continued).

$$= \sum_{k=1}^{\infty} \inf_k \sum_{i=1}^{\infty} \left(\mathrm{diam}(A_i^k)\right)^{\alpha} \text{ where } \inf_k \text{ denotes an infimum over all } \{A_i^k\}_{i=1}^{\infty} \text{ coverings of } E_k$$

$$= \sum_{k=1}^{\infty} H_\alpha^{(\varepsilon)}(E_k).$$

Now since $\{A_i^k\}_{i,k=1}^{\infty}$ is *some* covering of E (namely, one based on a union of coverings of the E_k) then when an infimum is taken over all coverings of E we have $H_\alpha^{(\varepsilon)}(E) \leq \inf \sum_{i,k=1}^{\infty} (\mathrm{diam}(A_i^k))^{\alpha}$ and hence $H_\alpha^{(\varepsilon)}(E) \leq \sum_{k=1}^{\infty} H_\alpha^{(\varepsilon)}(E_k)$. a limit as $\varepsilon \to 0$ we have $H_\alpha^*(E) \leq \sum_{k=1}^{\infty} H_\alpha^*(E_k)$ so that H_α^* is countably monotone. So H_α^* is (by definition) an outer measure on 2^X.

Proposition 20.29 (continued 2)

Proof (continued). To establish that H_α^* is a Carathéodory outer measure, let $E, F \subset X$ for which $\rho(E,F) > \delta > 0$. Let $\varepsilon > 0$ be such that $\varepsilon < \delta$. If $\{A_k\}_{k=1}^{\infty}$ is a cover of $E \cup F$ then since $\varepsilon < \delta$ then each A_k can intersect at most one of E and F. So any such cover of $E \cup F$ yields a cover of E and a cover of F. Taking an infimum over all such coverings of $E \cup F$ we have

$$H_\alpha^{(\varepsilon)}(E \cup F) = \inf \sum_{k=1}^{\infty} (\mathrm{diam}(A_k))^{\alpha}$$

$$\geq \inf_E \sum_{k=1}^{\infty} \left(\mathrm{diam}(A_k^E)\right)^{\alpha} + \inf_F \sum_{k=1}^{\infty} \left(\mathrm{diam}(A_k^E)\right)^{\alpha} = H_\alpha^{(\varepsilon)}(E) + H_\alpha^{(\varepsilon)}(F)$$

where $\inf_E$ is an infimum over all coverings $\{A_k^E\}_{k=1}^{\infty}$ of E (and similarly for $\inf_F$); the inequality is introduced since (for $\varepsilon < \delta$) every covering $E \cup F$ implies a covering of E and a covering of F but $\inf_E$ and $\inf_F$ involves *more* potential coverings of E and F.

Proposition 20.29 (continued 3)

Proposition 20.29. Let (X, ρ) be a metric space and α a positive real number. Then $H_\alpha^* : 2^X \to [0, \infty]$ is a Carathéodory outer measure.

Proof (continued). Taking a limit as $\varepsilon \to 0$ we get $H_\alpha^*(E \cup F) \geq H_\alpha^*(E) + H_\alpha^*(F)$. We showed above that H_α^* is countably monotone and so $H_\alpha^*(E \cup F) \leq H_\alpha^*(E) + H_\alpha^*(F)$ so that H_α^* is (by definition) a Carathéodory outer measure. □

Proposition 20.30

Proposition 20.30. Let (X, ρ) be a metric space. Let A be a Borel subset of X, and let α, β be positive real numbers for which $\alpha < \beta$. If $H_\alpha(A) < \infty$ then $H_\beta(A) = 0$.

Proof. Let $\varepsilon > 0$. Choose $\{A_k\}_{k=1}^{\infty}$ as a covering of A by sets of diameter less than or equal to ε for which

$$\sum_{k=1}^{\infty} (\mathrm{diam}(A_k))^{\alpha} \leq H_\alpha^*(A) + 1 = H_\alpha(A) + 1$$

(which can be done, by the definition of "infimum"). Then

$$H_\beta^{(\varepsilon)}(A) \leq \sum_{k=1}^{\infty} (\mathrm{diam}(A_k))^{\beta} \text{ by the infimum definition of } H_\beta^{(\varepsilon)}(A)$$

$$= \varepsilon^{\beta} \sum_{k=1}^{\infty} \left(\frac{\mathrm{diam}(A_k)}{\varepsilon}\right)^{\alpha}$$

Proposition 20.30 (continued 1)

Proof (continued).

$$\leq \varepsilon^{\beta} \sum_{k=1}^{\infty} \left(\frac{\text{diam}(A_k)}{\varepsilon} \right)^{\alpha} \text{ since } \text{diam}(A_k) \leq \varepsilon, \frac{\text{diam}(A_k)}{\varepsilon} \leq 1 \text{ and so}$$

$$\left(\frac{\text{diam}(A_k)}{\varepsilon} \right)^{\beta} \leq \left(\frac{\text{diam}(A_k)}{\varepsilon} \right)^{\alpha} \text{ because } \alpha < \beta$$

$$= \varepsilon^{\beta-\alpha} \sum_{k=1}^{\infty} (\text{diam}(A_k))^{\alpha} \leq \varepsilon^{\beta-\alpha}(H_{\alpha}(A) + 1.$$

Taking a limit as $\varepsilon \to 0$ we get

$$H_{\beta}^{*}(A) = \lim_{\varepsilon \to 0} H_{\beta}^{(\varepsilon)}(A) \leq \lim_{\varepsilon \to 0} \varepsilon^{\beta-\alpha}(H_{\alpha}(A) + 1) = 0$$

since $\beta - \alpha > 0$ and $H_{\alpha}(A) + 1$ is finite. Since $H_{\beta}^{*}(A) = 0$ (here, H_{β}^{*} is a Carathéodory outer measure) then the induced Hausdorff β-dimensional measure satisfies $H_{\beta}(A) = 0$. □

Theorem 20.4.A

Theorem 20.4.A. The Hausdorff 1-dimensional measure, H_1, is the same as Lebesgue measure on the σ-algebra of Lebesgue measurable sets of real numbers.

Proof. Let $I \subset \mathbb{R}$ be an interval. Given $\varepsilon > 0$, I can be expressed as the disjoint union of subintervals of length less than ε and the diameter of each subinterval is its length (we make no restriction on any of the intervals in terms of open/closed). So

$$H_1^{(\varepsilon)}(I) = \inf \sum_{k=1}^{\infty} \ell(I_k) = m^{*}(I) = m(I) = \ell(I)$$

and so $H_1(I) = \lim_{\varepsilon \to 0} H_1^{(\varepsilon)}(I) = m^{*}(I)$. Thus H_1 and Lebesgue measure agree on the semiring of intervals of real numbers. Since H_1 and Lebesgue measure are extensions of the same premeasure on the semiring of intervals, then by the uniqueness claim of the Carathéodory-Hahn Theorem, Lebesgue measure and H_1 are equal on the σ-algebra of measurable sets. □