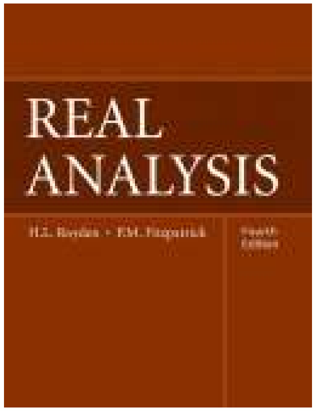
Real Analysis

Chapter 22. Invariant Measures

22.1. Toplogical Groups: The General Linear Group—Proofs of Theorems



Lemma 22.1

Lemma 22.1. Let E be a Banach space and the operator $C \in \mathcal{L}(E)$ have $\|C\| < 1$. Then $\mathrm{Id} - c$ is invertible and $\|(\mathrm{Id} - C)^{-1}\| \le (1 - \|C\|)^{-1}$.

Proof. By Exercise 22.7, the series $\sum_{k=0}^{\infty} C^k$ converges in $\mathcal{L}(E)$. So $\sum_{k=0}^{\infty} C^k$ converges to a continuous (that is, bounded) linear operator. But

$$(\mathrm{Id} - C) \circ \left(\sum_{k=0}^{\infty} C^k \right) = \left(\sum_{k=0}^{\infty} C^k \right) \circ (\mathrm{Id} - C) = \mathrm{Id} - C^{n+1}$$

for all $n \in \mathbb{N}$ (notice that C^k commutes with Id and with C). Since $\|C^k\| \to 0$ because $\|C^k\| \le \|C\|^k$ (by the first Note on page 3 of the class notes or, by (1) on page 478) and $\|C\|^k \to 0$. So $C^k \to 0$ in $\mathcal{L}(E)$. So taking a limit on the right side of the equation above gives Id, and hence the limit of the left hand side must also be Id. This implies that $\sum_{k=0}^{\infty} C^k = (\mathrm{Id} - C)^{-1}$. So

$$\|(\mathrm{Id} - C)^{-1}\| = \left\| \sum_{k=0}^{\infty} C^k \right\| \le \sum_{k=0}^{\infty} \|C^k\| \le \sum_{k=0}^{\infty} \|C\|^k = \frac{1}{1 - \|C\|}. \quad \square$$

Theorem 22.2

Theorem 22.2. Let E be a Banach space. Then the general linear group of E, $GL(E)$, is a topological group with respect to the group operator of operator composition and the topology induced by the operator norm on $\mathcal{L}(E)$.

Proof. We observed above that $GL(E)$ is a group. Since the topology is induced by the operator norm, it is metrizable and so is Hausdorff (by Proposition 11.7). So we need only show that the binary operation (namely, operator composition) is continuous in the product topology and that inversion is continuous.

First, we show continuity of operator composition. For $T, T', S, S' \in GL(E)$ we have

$$T \circ S - T' \circ S' - T \circ S - T \circ S' + T \circ S' - T' \circ S' - T \circ (S - S') + (T - T') \circ S'$$

since T and S' are linear.

Theorem 22.2 (continued 1)

Proof (continued). So

$$\begin{aligned} \|T \circ S - T' \circ S'\| &= \|T \circ (S - S') + (T - T') \circ S'\| \\ &\le \|T \circ (S - S')\| + \|(T - T') \circ S'\| \\ &\quad \text{by the Triangle Inequality for the operator norm} \\ &\le \|T\|\|S - S'\| + \|T - T'\|\|S\| \\ &\quad \text{since } \|T \circ S\| \le \|T\|\|S\| \text{ in general.} \end{aligned}$$

Let $\varepsilon > 0$. Think of $(S, T) \in \mathcal{G} \times \mathcal{G}$ as fixed. Notice $0 \notin GL(E)$ since 0 is not invertible and so $T \ne 0$. Let $\delta_1 = \frac{\varepsilon}{2\|T\|}$ and $\delta_2 = \frac{\varepsilon}{2(\|S\| + \delta_1)} = \frac{\|T\|\varepsilon}{2\|S\| + \|T\|\varepsilon}$. Consider the open set in the product topology for $\mathcal{G} \times \mathcal{G}$ of $\mathcal{O} = \{S' \mid \|S - S'\| < \delta_1\} \times \|T' \mid \|T - T'\| < \delta_2\}$. For all $(S', T') \in \mathcal{O}$ we have

$$\|T\|\|S - S'\| < \|T\|\delta_1 = \|T\| \frac{\varepsilon}{2\|T\|} = \frac{\varepsilon}{2} \text{ and}$$

Theorem 22.2 (continued 2)

Proof (continued).

$$\begin{aligned}\|T - T'\|\|S'\| &< \|S'\|\delta_2 < (\|S\| + \delta_1) \text{ since } \|S - S'\| < \delta_1 \\ &< (\|S\| + \delta_1)\frac{\varepsilon}{2(\|S\| + \delta_1)} = \frac{\varepsilon}{2}.\end{aligned}$$

So

$$\|T \circ S - T' \circ S'\| \le \|T\|\|S - S'\| + \|T - T'\|\|S'\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore operator composition is continuous.

Second, for continuity of inversion. If $S \in GL(E)$ and $\|S - \mathrm{Id}\| < 1$, then the identity $S^{-1} - \mathrm{Id} = (\mathrm{Id} - S) \circ S^{-1} = (\mathrm{Id} - S) \circ (\mathrm{Id} - (\mathrm{Id} - S))^{-1}$ gives

$$\begin{aligned}\|S^{-1} - \mathrm{Id}\| &= \|(\mathrm{Id} - S) \circ (\mathrm{Id} - (\mathrm{Id} - S))^{-1}\| \\ &\le \|\mathrm{Id} - S\|\|(\mathrm{Id} - (\mathrm{Id} - S))^{-1}\| \text{ since } \|S \circ T\| \le \|S\|\|T\| \\ &\le \frac{\|S - \mathrm{Id}\|}{1 - \|S - \mathrm{Id}\|} \text{ by Lemma 22.1.} \qquad (3)\end{aligned}$$

Theorem 22.2 (continued 3)

Proof (continued). Let $\varepsilon > 0$. Let $\delta = \min\{1/2, \varepsilon/2\}$. Then for any S in the open set $\mathcal{O} = \{S \mid \|S - \mathrm{Id}\| < \delta\}$ we have $a - \|S - \mathrm{Id}\| > 1 - 1/2 = 1/2$ and so $\frac{1}{a - \|S - \mathrm{Id}\|} < 2$. Therefore for all $S \in \mathcal{O}$ we have

$$\frac{\|S - \mathrm{Id}\|}{1 - \|S - \mathrm{Id}\|} < 2\|S - \mathrm{Id}\| \le c\frac{\varepsilon}{2} = \varepsilon.$$

Hence inversion is continuous at Id. Now let $T \in GL(E)$. For any $S \in GL(E)$ with $\|S - T\| < \|T^{-1}\|^{-1}$ we have

$$\begin{aligned}\|T^{-1} \circ S - \mathrm{Id}\| &= \|T^{-1} \circ (S - T)\| \text{ since } T^{-1} \text{ is linear} \\ &\le \|T^{-1}\|\|S - T\| \text{ since } \|T \circ S\| \le \|S\|\|T\| \text{ in general} \\ &< 1.\end{aligned}$$

Replacing S in (3) with $T^{-1} \circ S$ we have

$$\|S^{-1} \circ T - Id\| \le \frac{\|T^{-1} \circ S - \mathrm{Id}\|}{1 - \|T^{-1} \circ S - \mathrm{Id}\|}.$$

Theorem 22.2 (continued 4)

Proof (continued). Since $S^{-1} - T^{-1} = (S^{-1} \circ T - \mathrm{Id}) \circ T^{-1}$ and $T^{-1} \circ S - \mathrm{Id} = T^{-1} \circ (S - T)$ we then have

$$\|S^{-1} - T^{-1}\| = \|(S^{-1} \circ T - \mathrm{Id}) \circ T^{-1}\| \le \|S^{-1} \circ T - \mathrm{Id}\|\|T^{-1}\|$$

$$\le \frac{\|T^{-1} \circ S - \mathrm{Id}\|}{1 - \|T^{-1} \circ S - \mathrm{Id}\|}\|T^{-1}\| = \frac{\|T^{-1} \circ (S - T)\|\|T^{-1}\|}{1 - \|T^{-1} \circ (S - T)\|}$$

$$\le \frac{\|T^{-1}\|^2\|T - S\|}{1 - \|T^{-1}\|\|T - S\|}.$$

Let $\varepsilon > 0$. Let

$$\delta = \min\left\{\frac{1}{2\|T^{-1}\|}, \frac{\varepsilon}{2\|T^{-1}\|^2}\right\}.$$

Then for any S in the open set $\mathcal{O} = \{S \mid \|T - S\| < \delta\}$ we have

$$1 - \|T^{-1}\|\|T - S\| > 1 - \frac{\|T^{-1}\|}{2\|T^{-1}\|} = \frac{1}{2}$$

Theorem 22.2 (continued 5)

Theorem 22.2. Let E be a Banach space. Then the general linear group of E, $GL(E)$, is a topological group with respect to the group operator of operator composition and the topology induced by the operator norm on $\mathcal{L}(E)$.

Proof (continued). …and so

$$\frac{1}{1 - \|T^{-1}\|\|T - S\|} < 2.$$

Therefore for all $S \in \mathcal{O}$ we have

$$\|S^{-1} - T^{-1}\| \le \frac{\|T^{-1}\|^2\|T - S\|}{1 - \|T^{-1}\|\|T - S\|} < 2\|T^{-1}\|^2\|T - S\|$$

$$< 2\|T^{-1}\|^2\frac{\varepsilon}{2\|T^{-1}\|^2} = \varepsilon.$$

Therefore, inversion is continuous at arbitrary $T \in GL(E)$ and hence inversion is continuous on $GL(E)$. Therefore $GL(E)$ is a topological group. □