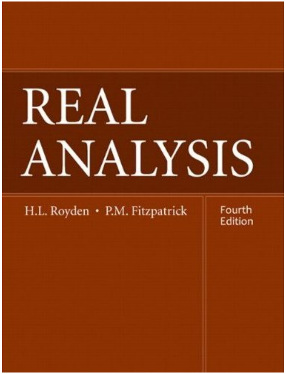
Real Analysis

Chapter 22. Invariant Measures

22.1. Toplogical Groups: The General Linear Group—Proofs of Theorems



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$$(\mathrm{Id} - C) \circ \left(\sum_{k=0}^{\infty} C^k \right) = \left(\sum_{k=0}^{\infty} C^k \right) \circ (\mathrm{Id} - C) = \mathrm{Id} - C^{n+1}$$

for all $n \in \mathbb{N}$ (notice that C^k commutes with Id and with C). Since $\|C^k\| \to 0$ because $\|C^k\| \leq \|C\|^k$ (by the first Note on page 3 of the class notes or, by (1) on page 478) and $\|C\|^k \to 0$. So $C^k \to 0$ in $\mathcal{L}(E)$.

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Theorem 22.2

Theorem 22.2. Let E be a Banach space. Then the general linear group of E, $GL(E)$, is a topological group with respect to the group operator of operator composition and the topology induced by the operator norm on $\mathcal{L}(E)$.

Proof. We observed above that $GL(E)$ is a group. Since the topology is induced by the operator norm, it is metrizable and so is Hausdorff (by Proposition 11.7). So we need only show that the binary operation (namely, operator composition) is continuous in the product topology and that inversion is continuous.

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First, we show continuity of operator composition. For $T, T', S, S' \in GL(E)$ we have

$$T \circ S - T' \circ S' - T \circ S - T \circ S' + T \circ S' - T' \circ S' - T \circ (S - S') + (T - T') \circ S'$$

since T and S' are linear.

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Theorem 22.2 (continued 1)

Proof (continued). So

$$\begin{aligned}\|T \circ S - T' \circ S'\| &= \|T \circ (S - S') + (T - T') \circ S'\| \\ &\leq \|T \circ (S - S')\| + \|(T - T') \circ S'\| \\ &\quad \text{by the Triangle Inequality for the operator norm} \\ &\leq \|T\|\|S - S'\| + \|T - T'\|\|S\| \\ &\quad \text{since } \|T \circ S\| \leq \|T\|\|S\| \text{ in general.}\end{aligned}$$

Let $\varepsilon > 0$. Think of $(S, T) \in \mathcal{G} \times \mathcal{G}$ as fixed. Notice $0 \notin GL(E)$ since 0 is not invertible and so $T \neq 0$.

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$$\|T\|\|S - S'\| < \|T\|\delta_1 = \|T\|\frac{\varepsilon}{2\|T\|} = \frac{\varepsilon}{2} \text{ and}$$

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Proof (continued).

$$\begin{aligned} \|T - T'\|\|S'\| &< \|S'\|\delta_2 < (\|S\| + \delta_1) \text{ since } \|S - S'\| < \delta_1 \\ &< (\|S\| + \delta_1)\frac{\varepsilon}{2(\|S\| + \delta_1)} = \frac{\varepsilon}{2}. \end{aligned}$$

So

$$\|T \circ S - T' \circ S'\| \leq \|T\|\|S - S'\| + \|T - T'\|\|S'\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore operator composition is continuous.

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Second, for continuity of inversion. If $S \in GL(E)$ and $\|S - \mathrm{Id}\| < 1$, then the identity $S^{-1} - \mathrm{Id} = (\mathrm{Id} - S) \circ S^{-1} = (\mathrm{Id} - S) \circ (\mathrm{Id} - (\mathrm{Id} - S))^{-1}$ gives

$$\begin{aligned} \|S^{-1} - \mathrm{Id}\| &= \|(\mathrm{Id} - S) \circ (\mathrm{Id} - (\mathrm{Id} - S))^{-1}\| \\ &\leq \|\mathrm{Id} - S\|\|(\mathrm{Id} - (\mathrm{Id} - S))^{-1}\| \text{ since } \|S \circ T\| \leq \|S\|\|T\| \\ &\leq \frac{\|S - \mathrm{Id}\|}{1 - \|S - \mathrm{Id}\|} \text{ by Lemma 22.1.} \qquad (3) \end{aligned}$$

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Proof (continued). Let $\varepsilon > 0$. Let $\delta = \min\{1/2, \varepsilon/2\}$. Then for any S in the open set $\mathcal{O} = \{S \mid \|S - \mathrm{Id}\| < \delta\}$ we have $a - \|S - \mathrm{Id}\| > 1 - 1/2 = 1/2$ and so $\frac{1}{a - \|S - \mathrm{Id}\|} < 2$. Therefore for all $S \in \mathcal{O}$ we have

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Proof (continued). Since $S^{-1} - T^{-1} = (S^{-1} \circ T - \mathrm{Id}) \circ T^{-1}$ and $T^{-1} \circ S - \mathrm{Id} = T^{-1} \circ (S - T)$ we then have

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Therefore for all $S \in \mathcal{O}$ we have

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Therefore, inversion is continuous at arbitrary $T \in GL(E)$ and hence inversion is continuous on $GL(E)$. Therefore $GL(E)$ is a topological group. □

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