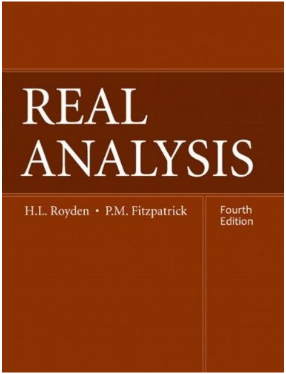
Real Analysis

Chapter 3. Lebesgue Measurable Functions

3.1. Sums, Products, and Compositions—Proofs of Theorems



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(i) For each $c \in \mathbb{R}$, $\{x \in E \mid f(x) > c\} \in \mathcal{M}$.

(ii) For each $c \in \mathbb{R}$, $\{x \in E \mid f(x) \geq c\} \in \mathcal{M}$.

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Each of these properties implies that for each extended real number c, $\{x \in E \mid f(x) = c\} \in \mathcal{M}$.

Proof. The sets in (i) and (iv) are complements and the sets in (ii) and (iii) are complements. Since the complements of measurable sets are measurable, then (i) and (iv) are equivalent and (ii) and (iii) are equivalent.

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Proof. The sets in (i) and (iv) are complements and the sets in (ii) and (iii) are complements. Since the complements of measurable sets are measurable, then (i) and (iv) are equivalent and (ii) and (iii) are equivalent. Since $\{x \in E \mid f(x) \geq c\} = \cap_{k=1}^{\infty}\{x \in E \mid f(x) > c - 1/k\}$, then (i) implies (ii). Since $\{x \in E \mid f(x) > c\} = \cup_{k=1}^{\infty}\{x \in E \mid f(x) \geq c + 1/k\}$, then (ii) implies (i). So (i), (ii), (iii), and (iv) are equivalent.

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Proposition 3.2

Proposition 3.2. Let f be defined on $E \in \mathcal{M}$. Then f is measurable if and only if for each open $\mathcal{O}$, the inverse image of $\mathcal{O}$, $f^{-1}(\mathcal{O})$, is measurable.

Proof. First, if the inverse image under f of every open set is measurable, then for every $c \in \mathbb{R}$ we have that the inverse image of (c, ∞), $f^{-1}((c, \infty)) = \{x \in E \mid f(x) > c\}$, is measurable. Therefore, by Proposition 3.1(i), f is measurable.

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Second, suppose f is measurable and let $\mathcal{O}$ be open. Then $\mathcal{O}$ is a countable disjoint union of open intervals. Now any unbounded open interval can be written as a countable union of bounded open intervals, so $\mathcal{O} = \cup_{k=1}^{\infty} I_k$ where each I_k is bounded, say $I_k = (a_k, b_k) = B_k \cap A_k$ where $B_k = (-\infty, b_k)$ and $A_k = (a_k, \infty)$. Since f is a measurable function, each $f^{-1}(B_k)$ and $f^{-1}(A_k)$ are measurable sets by Proposition 3.1(i and iii).

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Proof (continued). Since

$$f^{-1}(\mathcal{O}) = f^{-1}\left(\cup_{k=1}^{\infty} B_k \cap A_k\right)$$

$$= \cup_{k=1}^{\infty} f^{-1}(B_k \cap A_k) = \cup_{k=1}^{\infty}(f^{-1}(B_k) \cap f^{-1}(A_k))$$

and since the measurable sets form a σ-algebra by Note 2.3.B, then $f^{-1}(\mathcal{O})$ is measurable, as claimed. $\square$

Proposition 3.3

Proposition 3.3. A real-valued function that is continuous on its measurable domain is measurable.

Proof. For continuous f, we have for all open $\mathcal{O}$ that $f^{-1}(\mathcal{O})$ is open relative to the domain of f (see my online notes for Analysis 1 [MATH 4217/5217] on 4.1. Limits and Continuity; see Theorem 4.5). So for $\mathcal{O}$ open, $f^{-1}(\mathcal{O}) = E \cap \mathcal{U}$ where $\mathcal{U}$ is open (i.e., $f^{-1}(\mathcal{O})$ is open with respect to E), and so $f^{-1}(\mathcal{O})$ is measurable. By Proposition 3.2 we now have that f is a measurable function, as claimed. □

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Proposition 3.5

Proposition 3.5. Let f be an extended real-valued function defined on E.

(i) If f is measurable on E and $f = g$ a.e., then g is measurable on E.

(ii) For $D \subseteq E$, $D \in \mathcal{M}$, f is measurable on E if and only if f restricted to D is measurable and f restricted to $E \setminus D$ is measurable.

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Proof. (i) Suppose f is measurable on E (then $E \in \mathcal{M}$). Define $A = \{x \in E \mid f(x) \neq g(x)\}$. Since $f = g$ a.e. then $m(A) = 0$. Then $\{x \in E \mid g(x) > c\} = \{x \in A \mid g(x) > c\} \cup [\{x \in E \mid f(x) > c\} \cap (E \setminus A)]$ for all $c \in \mathbb{R}$. Since $m(A) = 0$, then $\{x \in A \mid g(x) > c\} \in \mathcal{M}$ by Proposition 2.4.

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Proposition 3.6(i)

Proposition 3.6. Suppose f, g are measurable on E and f, g are extended real-valued functions which are finite a.e. on E. Then

(i) $\alpha f + \beta g$ is measurable on E for all $\alpha, \beta \in \mathbb{R}$.

(ii) fg is measurable on E.

Proof of (i). By Proposition 3.5(i), we may assume f and g are finite on all of E.

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For $x \in E$, if $f(x) + g(x) < c$, then $f(x) < c - g(x)$ and so there is $q \in \mathbb{Q}$ such that $f(x) < q < c - g(x)$. Then $\{x \in E \mid f(x) + g(x) < c\} = \cup_{q \in \mathbb{Q}}[\{x \in E \mid g(x) < c - q\} \cap \{x \in E \mid f(x) < q\}]$. This set is measurable and so $f + g$ is measurable.

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Since scalar multiples of measurable functions are measurable and since sums of measurable functions are measurable, then linear combinations of measurable functions are measurable.

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For $x \in E$, if $f(x) + g(x) < c$, then $f(x) < c - g(x)$ and so there is $q \in \mathbb{Q}$ such that $f(x) < q < c - g(x)$. Then $\{x \in E \mid f(x) + g(x) < c\} = \cup_{q \in \mathbb{Q}}[\{x \in E \mid g(x) < c - q\} \cap \{x \in E \mid f(x) < q\}]$. This set is measurable and so $f + g$ is measurable.

Since scalar multiples of measurable functions are measurable and since sums of measurable functions are measurable, then linear combinations of measurable functions are measurable.

Proposition 3.6(ii)

Proof of (ii). To prove (ii), observe that for all $x \in E$, $f(x)g(x) = \frac{1}{2}[(f(x) + g(x))^2 - f(x)^2 - g(x)^2]$. For $c \in \mathbb{R}$ with $c \geq 0$ we have

$$\{x \in E \mid f(x)^2 > c\} = \{x \in E \mid f(x) > \sqrt{c}\} \cup \{x \in E \mid f(x) < -\sqrt{c}\}.$$

For $c \in \mathbb{R}$ with $c < 0$ we have

$$\{x \in E \mid f(x)^2 > c\} = E.$$

In both cases, since f is a measurable function, we see that f^2 is a measurable function.

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In both cases, since f is a measurable function, we see that f^2 is a measurable function. Similarly, g^2 and (since $f+g$ is measurable by (i)) $(f+g)^2$ are measurable functions. Since by part (i), linear combinations of measurable functions are measurable, then we now have that $fg = \frac{1}{2}[(f+g)^2 - f^2 - g^2]$ is measurable. $\square$

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Proposition 3.7

Proposition 3.7. Let g be a measurable real-valued function defined on E and f a continuous real-valued function defined on all of $\mathbb{R}$. The composition $f \circ g$ is measurable on E.

Proof. Let $\mathcal{O}$ be open. Then $(f \circ g)^{-1}(\mathcal{O}) = g^{-1}(f^{-1}(\mathcal{O}))$. Since f is continuous, $f^{-1}(\mathcal{O})$ is open.

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Proposition 3.8

Proposition 3.8. For a finite family $\{f_k\}_{k=1}^n$ of measurable functions with common domain E, the functions $\max\{f_1, f_2, \ldots, f_n\}$ and $\min\{f_1, f_2, \ldots, f_n\}$ (defined pointwise) are measurable.

Proof. For any $c \in \mathbb{R}$, we have

$$\{x \in E \mid \max_{1 \le k \le n}\{f_k\} > c\} = \cup_{k=1}^n \{x \in E \mid f_k(x) > c\} \text{ and}$$

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