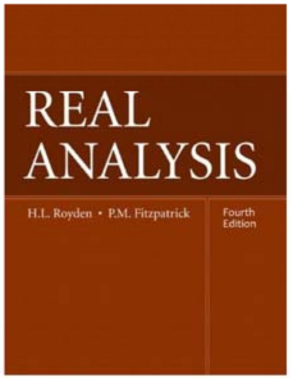
Real Analysis

Chapter 3. Lebesgue Measurable Functions

3.2. Sequential Pointwise Limits and Simple Approximation—Proofs of Theorems



Proposition 3.9

Proposition 3.9. Let $\{f_n\}$ be a sequence of measurable functions on E that converges pointwise a.e. on E to function f. Then f is measurable.

Proof. Let $E_0 \subset E$ for which $m(E_0) = 0$ and $\{f_n\}$ converges to f pointwise on $E \setminus E_0$. By Proposition 3.5(i), f is measurable if and only if its restriction to $E \setminus E_0$ is measurable. So, without loss of generality, we assume pointwise convergence on all of E.

Let $c \in \mathbb{R}$. For a given $x \in E$ we have $\lim_{n\to\infty} f_n(x) = f(x)$ and so $f(x) < c$ if and only if there exists $n, k \in \mathbb{N}$ for which $f_j(x) < c - 1/n$ for all $j \geq k$. Since f_j is measurable, then $\{x \in E \mid f_j(x) < c - 1/n\}$ is measurable for all $n, j \in \mathbb{N}$. So we have $\cap_{j=k}^{\infty}\{x \in E \mid f_j(x) < c - 1/n\} \in \mathcal{M}$. Finally, $\{x \in E \mid f(x) < c\} = \cup_{k,n=1}^{\infty}[\cap_{j=k}^{\infty}\{x \in E \mid f_j(x) < c - 1/n\}]$ is measurable and f is measurable. □

The Simple Approximation Lemma

The Simple Approximation Lemma.
Let f be measurable and real valued on set E. Assume f is bounded on E (i.e., $|f| \leq M$ on E for some M). Then for each $\varepsilon > 0$, there are simple functions φ_ε and ψ_ε for which

$$\varphi_\varepsilon \leq f \leq \psi_\varepsilon \text{ and } 0 \leq \psi_\varepsilon - \varphi_\varepsilon < \varepsilon \text{ on } E.$$

(That is, these inequalities hold pointwise for each $x \in E$.)

Proof. Let (c, d) be an open bounded interval that contains $f(E)$ and partition (c, d) as $c = y_0 < y_1 < \cdots < y_n = d$ such that $y_k - y_{k-1} < \varepsilon$ for each k. Define $I_k = [y_{k-1}, y_k)$ and $E_k = f^{-1}(I_k)$. Since f is measurable, each $E_k \in \mathcal{M}$. Define $\varphi_\varepsilon = \sum_{k=1}^{n} y_{k-1}\chi_{E_k}$ and $\psi_\varepsilon = \sum_{k=1}^{n} y_k\chi_{E_k}$. For each $x \in E$, $f(x) \in I_k$ for some k and so $\varphi_\varepsilon(x) = y_{k-1} \leq f(x) < y_k = \psi_\varepsilon(x)$. Since $y_k - y_{k-1} < \varepsilon$, then it follows that $0 < \psi_\varepsilon(x) - \varphi_\varepsilon(x) < y_k - y_{k-1}$ (for all k) $< \varepsilon$. □

The Simple Approximation Theorem

The Simple Approximation Theorem.
An extended real-valued function f on a measurable set E is measurable if and only if there is a sequence $\{\varphi_n\}$ of simple functions on E which converges pointwise on E to f and has the property that $|\varphi_n| \leq |f|$ on E for all n. If f is nonnegative, we may choose $\{\varphi_n\}$ to be increasing.

Proof. First, suppose that the sequence $\{\varphi_n\}$ of simple functions on E exists as described. Each simple function is measurable (by definition of "simple function"), so by Proposition 3.9 the pointwise limit of $\{\varphi_n\}$ is measurable. That is, f is measurable.

Second, suppose f is measurable. We also assume $f \geq 0$ on E. The general case will then follow by expressing f as the difference of nonnegative measurable functions, as shown in Problem 3.23. Let $n \in \mathbb{N}$. Define $E_n = \{x \in E \mid f(x) \leq n\}$. Then E_n is a measurable set (by definition of "measurable function") and the restriction of f to E_n is a nonnegative bounded measurable function (measurable by Proposition 3.5(ii)).

The Simple Approximation Theorem (continued 1)

Proof (continued). By the Simple Approximation Lemma, applied to the restriction of f to E_n and with $\varepsilon = 1/n$, we may select simple functions φ_n and ψ_n defined on E_n which satisfy

$$0 \le \varphi_n \le f \le \psi_n \text{ on } E_n \text{ and } 0 \le \psi_n - \varphi_n < 1/n \text{ on } E_n.$$

So $0 \le \varphi_n \le f$ and $0 \le f - \varphi_n \le \psi_n - \varphi_n < 1/n$ on E_n. Extend φ_n to all of E by setting $\varphi_n(x) = n$ if $f(x) \ge n$. Then the extended φ_n is a simple function defined on E and $0 \le \varphi_n \le f$ on E. We claim that the sequence φ_n converges to f pointwise on E. Let $x \in E$.

Case 1. Suppose $f(x)$ is finite. Choose $N \in \mathbb{N}$ for which $f(x) < N$. Then $0 \le f(x) - \varphi_n(x) < 1/n$ for $n \ge N$. Therefore $\lim_{n\to\infty} \varphi_n(x) = f(x)$.

Case 2. Suppose $f(x) = \infty$. Then $\varphi_n(x) = n$ for all n so that $\lim_{n\to\infty} \varphi_n(x) = \lim_{n\to\infty} n = \infty = f(x)$.

The Simple Approximation Theorem (continued 2)

The Simple Approximation Theorem.
An extended real-valued function f on a measurable set E is measurable if and only if there is a sequence $\{\varphi_n\}$ of simple functions on E which converges pointwise on E to f and has the property that $|\varphi_n| \le |f|$ on E for all n. If f is nonnegative, we may choose $\{\varphi_n\}$ to be increasing.

Proof (continued). If we now replace φ_n with $\max\{\varphi_1, \varphi_2, \ldots, \varphi_n\}$ (which is also simple by Problem 3.19) we have that the new sequence $\{\varphi_n\}$ is thus increasing and the new sequence is pointwise a subsequence of the original sequence and so converges to f pointwise as well. □