Proposition 3.9

Let \( \{f_n\} \) be a sequence of measurable functions on \( E \) that converges pointwise a.e. on \( E \) to function \( f \). Then \( f \) is measurable.

**Proof.** Let \( E_0 \subset E \) for which \( m(E_0) = 0 \) and \( \{f_n\} \) converges to \( f \) pointwise on \( E \setminus E_0 \). By Proposition 3.5(i), \( f \) is measurable if and only if its restriction to \( E \setminus E_0 \) is measurable. So, without loss of generality, we assume pointwise convergence on all of \( E \).

Let \( c \in \mathbb{R} \). For a given \( x \in E \) we have \( \lim_{n \to \infty} f_n(x) = f(x) \) and so \( f(x) < c \) if and only if there exists \( n, k \in \mathbb{N} \) for which \( f_j(x) < c - 1/n \) for all \( j \geq k \). Since \( f_j \) is measurable, then \( \{x \in E \mid f_j(x) < c - 1/n\} \) is measurable for all \( n, j \in \mathbb{N} \). So we have

\[
\cap_{j=k}^{\infty} \{x \in E \mid f_j(x) < c - 1/n\} \in \mathcal{M}.
\]

Finally,

\[
\{x \in E \mid f(x) < c\} = \bigcup_{k=1}^{\infty} \cap_{j=k}^{\infty} \{x \in E \mid f_j(x) < c - 1/n\}
\]

is measurable and \( f \) is measurable.

The Simple Approximation Lemma

Let \( f \) be measurable and real valued on set \( E \). Assume \( f \) is bounded on \( E \) (i.e., \( |f| \leq M \) on \( E \) for some \( M \)). Then for each \( \varepsilon > 0 \), there are simple functions \( \varphi_\varepsilon \) and \( \psi_\varepsilon \) for which

\[
\varphi_\varepsilon \leq f \leq \psi_\varepsilon \text{ and } 0 \leq \psi_\varepsilon - \varphi_\varepsilon < \varepsilon \text{ on } E.
\]

(That is, these inequalities hold pointwise for each \( x \in E \).)

**Proof.** Let \( (c, d) \) be an open bounded interval that contains \( f(E) \) and partition \( (c, d) \) as \( c = y_0 < y_1 < \cdots < y_n = d \) such that \( y_k - y_{k-1} < \varepsilon \) for each \( k \). Define \( I_k = (y_{k-1}, y_k) \) and \( E_k = f^{-1}(I_k) \). Since \( f \) is measurable, each \( E_k \in \mathcal{M} \). Define \( \varphi_\varepsilon = \sum_{k=1}^{n} y_{k-1} 1_{E_k} \) and \( \psi_\varepsilon = \sum_{k=1}^{n} y_k 1_{E_k} \). For each \( x \in E \), \( f(x) \in I_k \) for some \( k \) and so

\[
\varphi_\varepsilon(x) = y_{k-1} \leq f(x) < y_k = \psi_\varepsilon(x).
\]

Since \( y_k - y_{k-1} < \varepsilon \), then it follows that \( 0 < \psi_\varepsilon(x) - \varphi_\varepsilon(x) < y_k - y_{k-1} \) (for all \( k \)) < \varepsilon.

The Simple Approximation Theorem

An extended real-valued function \( f \) on a measurable set \( E \) is measurable if and only if there is a sequence \( \{\varphi_n\} \) of simple functions on \( E \) which converges pointwise on \( E \) to \( f \) and has the property that \( |\varphi_n| \leq |f| \) on \( E \) for all \( n \). If \( f \) is nonnegative, we may choose \( \{\varphi_n\} \) to be increasing.

**Proof.** First, suppose that the sequence \( \{\varphi_n\} \) of simple functions on \( E \) exists as described. Each simple function is measurable (by definition of “simple function”), so by Proposition 3.9 the pointwise limit of \( \{\varphi_n\} \) is measurable. That is, \( f \) is measurable.

Second, suppose \( f \) is measurable. We also assume \( f \geq 0 \) on \( E \). The general case will then follow by expressing \( f \) as the difference of nonnegative measurable functions, as shown in Problem 3.23. Let \( n \in \mathbb{N} \). Define \( F_n = \{x \in E \mid f(x) \leq n\} \). Then \( F_n \) is a measurable set (by definition of “measurable function”) and the restriction of \( f \) to \( E_n \) is a nonnegative bounded measurable function (measurable by Proposition 3.5(ii)).
The Simple Approximation Theorem (continued 1)

Proof (continued). By the Simple Approximation Lemma, applied to the restriction of $f$ to $E_n$ and with $\varepsilon = 1/n$, we may select simple functions $\varphi_n$ and $\psi_n$ defined on $E_n$ which satisfy

$$0 \leq \varphi_n \leq f \leq \psi_n \text{ on } E_n \text{ and } 0 \leq \psi_n - \varphi_n < 1/n \text{ on } E_n.$$ 

So $0 \leq \varphi_n \leq f$ and $0 \leq f - \varphi_n \leq \psi_n - \varphi_n < 1/n$ on $E_n$. Extend $\varphi_n$ to all of $E$ by setting $\varphi_n(x) = n$ if $f(x) \geq n$. Then the extended $\varphi_n$ is a simple function defined on $E$ and $0 \leq \varphi_n \leq f$ on $E$. We claim that the sequence $\varphi_n$ converges to $f$ pointwise on $E$. Let $x \in E$.

Case 1. Suppose $f(x)$ is finite. Choose $N \in \mathbb{N}$ for which $f(x) < N$. Then $0 \leq f(x) - \varphi_n(x) < 1/n$ for $n \geq N$. Therefore $\lim_{n \to \infty} \varphi_n(x) = f(x)$.

Case 2. Suppose $f(x) = \infty$. Then $\varphi_n(x) = n$ for all $n$ so that $\lim_{n \to \infty} \varphi_n(x) = \lim_{n \to \infty} n = \infty = f(x)$.

The Simple Approximation Theorem (continued 2)

An extended real-valued function $f$ on a measurable set $E$ is measurable if and only if there is a sequence $\{\varphi_n\}$ of simple functions on $E$ which converges pointwise on $E$ to $f$ and has the property that $|\varphi_n| \leq |f|$ on $E$ for all $n$. If $f$ is nonnegative, we may choose $\{\varphi_n\}$ to be increasing.

Proof (continued). If we now replace $\varphi_n$ with $\max\{\varphi_1, \varphi_2, \ldots, \varphi_n\}$ (which is also simple by Problem 3.19) we have that the new sequence $\{\varphi_n\}$ is thus increasing and the new sequence is pointwise a subsequence of the original sequence and so converges to $f$ pointwise as well. \qed