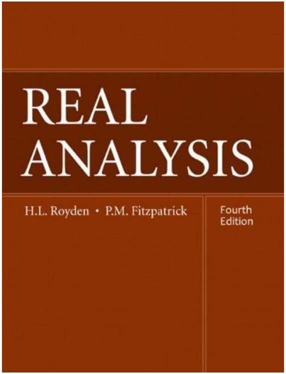
Real Analysis

Chapter 3. Lebesgue Measurable Functions

3.2. Sequential Pointwise Limits and Simple Approximation—Proofs of Theorems



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The Simple Approximation Lemma

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Let f be measurable and real valued on set E. Assume f is bounded on E (i.e., $|f| \leq M$ on E for some M). Then for each $\varepsilon > 0$, there are simple functions φ_ε and ψ_ε for which

$$\varphi_\varepsilon \leq f \leq \psi_\varepsilon \text{ and } 0 \leq \psi_\varepsilon - \varphi_\varepsilon < \varepsilon \text{ on } E.$$

(That is, these inequalities hold pointwise for each $x \in E$.)

Proof. Let (c, d) be an open bounded interval that contains $f(E)$ and partition (c, d) as $c = y_0 < y_1 < \cdots < y_n = d$ such that $y_k - y_{k-1} < \varepsilon$ for each k.

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Proof (continued). By the Simple Approximation Lemma, applied to the restriction of f to E_n and with $\varepsilon = 1/n$, we may select simple functions φ_n and ψ_n defined on E_n which satisfy

$$0 \leq \varphi_n \leq f \leq \psi_n \text{ on } E_n \text{ and } 0 \leq \psi_n - \varphi_n < 1/n \text{ on } E_n.$$

So $0 \leq \varphi_n \leq f$ and $0 \leq f - \varphi_n \leq \psi_n - \varphi_n < 1/n$ on E_n. Extend φ_n to all of E by setting $\varphi_n(x) = n$ if $f(x) \geq n$. Then the extended φ_n is a simple function defined on E and $0 \leq \varphi_n \leq f$ on E. We claim that the sequence φ_n converges to f pointwise on E. Let $x \in E$.

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Case 1. Suppose $f(x)$ is finite. Choose $N \in \mathbb{N}$ for which $f(x) < N$. Then $0 \leq f(x) - \varphi_n(x) < 1/n$ for $n \geq N$. Therefore $\lim_{n\to\infty} \varphi_n(x) = f(x)$.

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Case 2. Suppose $f(x) = \infty$. Then $\varphi_n(x) = n$ for all n so that $\lim_{n\to\infty} \varphi_n(x) = \lim_{n\to\infty} n = \infty = f(x)$.

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