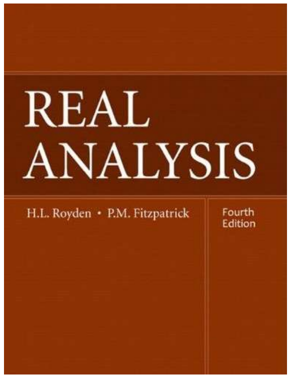
Real Analysis

Chapter 3. Lebesgue Measurable Functions

3.3. Littlewoods Three Principles, Egoroffs Theorem, and Lusins Theorem—Proofs of Theorems



Lemma 3.10

Lemma 3.10. Assume E has finite measure. Let $\{f_n\}$ be a sequence of measurable functions on E that converges pointwise on E to the real-valued function f. Then for each $\eta > 0$ and $\delta > 0$, there is a measurable subset A of E and an index N for which

$$|f_n - f| < \eta \text{ on } A \text{ for all } n \geq N \text{ and } m(E \setminus A) < \delta.$$

Proof. For each k, the function $|f - f_k|$ is well-defined (since f is real-valued then we do not have $\infty - \infty$ concerns, even though f_k might be extended real valued) and measurable by Theorem 3.6 and Proposition 3.9. So $\{x \in E \mid |f(x) - f_k(x)| < \eta\}$ is measurable for all $\eta \in \mathbb{R}$. Now $E_n = \{x \in E \mid |f(x) - f_k(x)| < \eta \text{ for all } k \geq n\} = \cap_{k=n}^{\infty}\{x \in E \mid |f(x) - f_k(x)| < \eta\}$ is measurable. Also, $\{E_n\}_{n=1}^{\infty}$ is an ascending collection of measurable sets. Since $\{f_n\}$ converges pointwise to f on E then $E = \cup_{n=1}^{\infty} E_n = \lim E_n$.

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Proof (continued). By continuity of measure (Theorem 2.15) $m(E) = \lim m(E_n)$. Since $m(E) < \infty$, we may choose $N \in \mathbb{N}$ such that $m(E_N) > m(E) - \delta$. Define $A = E_N$. Then by the Excision Property, $m(E \setminus A) = m(E) - m(A) = m(E) - m(E_N) < \delta$. □

Egoroff's Theorem

Egoroff's Theorem. Assume E has finite measure. Let $\{f_n\}$ be a sequence of measurable functions on E that converges pointwise on E to the real-valued function f. Then for each $\varepsilon > 0$, there is a closed set F contained in E for which

$$\{f_n\} \to f \text{ uniformly on } F \text{ and } m(E \setminus F) < \varepsilon.$$

Proof. Let $\varepsilon > 0$ and $n \in \mathbb{N}$. Let $\delta = \varepsilon/2^{n+1}$ and $\eta = 1/n$. Then by Lemma 3.10 (this is where finite measure is used) there exists measurable $A_n \subset E$ and $N(n) \in \mathbb{N}$ such that $|f_k - f| < \eta = 1/n$ on A_n for all $k \geq N(n)$ and $m(E \setminus A_n) < \delta = \varepsilon/2^{n+1}$. Define $A = \cap_{n=1}^{\infty} A_n$. Then

$$\begin{aligned} m(E \setminus A) &= m(E \setminus (\cap_{n=1}^{\infty} A_n)) \\ &= m(\cup_{n=1}^{\infty}(E \setminus A_n)) \text{ by DeMorgan's Laws} \\ &\leq \sum_{n=1}^{\infty} m(E \setminus A_n) \text{ by countable subadditivity} \end{aligned}$$

Egoroff's Theorem

Proof (continued).

$$m(E \setminus A) \leq \sum_{n=1}^{\infty} m(E \setminus A_n) < \sum_{n=1}^{\infty} \frac{\varepsilon}{2^{n+1}} = \frac{\varepsilon}{2}.$$

We now show that $\{f_n\} \to f$ uniformly on A. Let $\varepsilon > 0$ and choose n_0 such that $1/n_0 < \varepsilon$. Then from above there is $N(n_0) \in \mathbb{N}$ such that $|f_k - f| < 1/n_0$ on A_{n_0} for $k \geq N(n_0)$. Since $A \subset A_{n_0}$ and $1/n_0 < \varepsilon$ then the previous observation implies $|f_k - f| < \varepsilon$ on A for $k \geq N(n_0)$. So $\{f_n\}$ converges to f uniformly on A and $m(E \setminus A) < \varepsilon/2$.
Now we need to find the desired closed set. By Theorem 2.11 there is a closed set F contained in A for which $m(A \setminus F) < \varepsilon/2$. So $E \setminus F = (E \setminus A) \cup (A \setminus F)$ and $m(E \setminus F) = m(E \setminus A) + m(A \setminus F) < \varepsilon/2 + \varepsilon/2 = \varepsilon$. Since $F \subset A$, then $\{f_n\}$ converges uniformly on F. □

Proposition 3.11

Proposition 3.11. Let f be a simple function defined on E. Then for each $\varepsilon > 0$, there is a continuous function g on $\mathbb{R}$ and a closed set F contained in E for which

$$f = g \text{ on } F \text{ and } m(E \setminus F) < \varepsilon.$$

Proof. Let $a_1, a_2, \ldots, a_n$ be the finite number of distinct values taken by f and let the values be taken on the sets $E_1, E_2, \ldots, E_n$ respectively. Since the a_k's are distinct then the E_k's are disjoint. By Theorem 2.11 there are closed sets $F_1, F_2, \ldots, F_n$ such that for each k, $F_k \subset E_k$ and $m(E_k \setminus F_k) < \varepsilon/n$. Then $F = \cup_{k=1}^n F_k$ is closed. Since the E_k are disjoint, we have by countable additivity

$$m(E \setminus F) = m((\cup_{k=1}^n E_k) \setminus (\cup_{k=1}^n F_k)) = m(\cup_{k=1}^n (E_k \setminus F_k))$$

$$= \sum_{k=1}^{n} m(E_k \setminus F_k) < \sum_{k=1}^{n} \frac{\varepsilon}{n} = \varepsilon.$$

Proposition 3.11

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$$f = g \text{ on } F \text{ and } m(E \setminus F) < \varepsilon.$$

Proof (continued). Define g on F as $g(x) = a_k$ for $x \in F_k$. (The F_k's are disjoint, so g is well-defined.) Since the F_k's are closed, g is continuous on F (for $x \in F_k$, there is an open interval containing x which is disjoint from the other F_k's, so g is constant on this open interval intersecting F). By Problem 3.25, g can be extended to a function continuous on all of $\mathbb{R}$. This extension of g is the desired function. □

Lusin's Theorem

Lusin's Theorem. Let f be a real-valued measurable function on E. Then for each $\varepsilon > 0$, there is a continuous function g on $\mathbb{R}$ and a closed set F contained in E for which

$$f = g \text{ on } F \text{ and } m(E \setminus F) < \varepsilon.$$

Proof. The case $m(E) = \infty$ is Problem 3.29, so we consider $m(E) < \infty$. By the Simple Approximation Theorem, there is a sequence $\{f_n\}$ of simple functions defined on E that converges to f pointwise on E. Let $n \in \mathbb{N}$. By Proposition 3.11, with f replaced by f_n and ε replaced by $\varepsilon/2^{n+1}$, there is a continuous g_n defined on $\mathbb{R}$ and a closed set F_n contained in E for which $f_n = g_n$ on F_n and $m(E \setminus F_n) < \varepsilon/2^{n+1}$. By Egoroff's Theorem (this is where finite measure is used), there is a closed set F_0 contained in E such that $\{f_n\}$ converges to f uniformly on F_0 and $m(E \setminus F_0) < \varepsilon/2$. Define $F = \cap_{n=0}^{\infty} F_n$.

Lusin's Theorem

Proof (continued). Then

$$m(E\setminus F) = m(E\setminus\cap_{n=0}^{\infty}F_n) = m(\cup_{n=0}^{\infty}(E\setminus F_n)) = m((E\setminus F_0)\cup(\cup_{n=1}^{\infty}(E\setminus F_n)))$$

$$< \frac{\varepsilon}{2} + \sum_{n=1}^{\infty}\frac{\varepsilon}{2^{n+1}} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

The set F is closed (since it's the intersection of closed sets F_n). Each f_n is continuous on F since $F \subset F_n$ and $f_n = g_n$ on F_n and g_n is continuous on $\mathbb{R}$. Finally, $\{f_n\}$ converges to f uniformly on F since $F \subset F_0$ and $\{f_n\}$ converges uniformly to f on F_0 (that's how F_0 was chosen). However, the uniform limit of continuous functions is continuous, so the restriction of f to set F is continuous. By Problem 3.25, there is a continuous function g defined on all of $\mathbb{R}$ such that $g = f$ on F. Function g is the desired function. □