Chapter 3. Lebesgue Measurable Functions
3.3. Littlewood’s Three Principles, Egoroffs Theorem, and Lusins Theorem—Proofs of Theorems
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$$|f_n - f| < \eta \text{ on } A \text{ for all } n \geq N \text{ and } m(E \setminus A) < \delta.$$ 

Proof. For each $k$, the function $|f - f_k|$ is well-defined (since $f$ is real-valued then we do not have $\infty - \infty$ concerns, even though $f_k$ might be extended real valued) and measurable by Theorem 3.6 and Proposition 3.9. So $\{x \in E \mid |f(x) - f_k(x)| < \eta\}$ is measurable for all $\eta \in \mathbb{R}$. 
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$$E_n = \{x \in E \mid |f(x) - f_k(x)| < \eta \text{ for all } k \geq n\} = \bigcup_{k=n}^{\infty} \{x \in E \mid |f(x) - f_k(x)| < \eta\}$$

is measurable. Also, $\{E_n\}_{n=1}^{\infty}$ is an ascending collection of measurable sets. Since $\{f_n\}$ converges pointwise to $f$ on $E$ then $E = \bigcup_{n=1}^{\infty} E_n = \lim E_n$. 


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Proof (continued). By continuity of measure (Theorem 2.15) $m(E) = \lim m(E_n)$. 
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\[\Box\]
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$m(E \setminus A) = m(E) - m(E_N) < \delta$. \qed
Egoroff’s Theorem

**Egoroff’s Theorem.** Assume $E$ has finite measure. Let $\{f_n\}$ be a sequence of measurable functions on $E$ that converges pointwise on $E$ to the real-valued function $f$. Then for each $\varepsilon > 0$, there is a closed set $F$ contained in $E$ for which

$$\{f_n\} \to f \text{ uniformly on } F \text{ and } m(E \setminus F) < \varepsilon.$$ 

**Proof.** Let $\varepsilon > 0$ and $n \in \mathbb{N}$. Let $\delta = \varepsilon/2^{n+1}$ and $\eta = 1/n$. 
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$$m(E \setminus A) = m(E \setminus (\cap_{n=1}^{\infty} A_n))$$

$$= m(\cup_{n=1}^{\infty} (E \setminus A_n)) \text{ by DeMorgan’s Laws}$$

$$\leq \sum_{n=1}^{\infty} m(E \setminus A_n) \text{ by countable subadditivity}$$

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Proof (continued).

\[ m(E \setminus A) \leq \sum_{n=1}^{\infty} m(E \setminus A_n) < \sum_{n=1}^{\infty} \frac{\varepsilon}{2^{n+1}} = \frac{\varepsilon}{2}. \]

We now show that \( \{f_n\} \to f \) uniformly on \( A \). Let \( \varepsilon > 0 \) and choose \( n_0 \) such that \( 1/n_0 < \varepsilon \).
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We now show that \( \{f_n\} \to f \) uniformly on \( A \). Let \( \varepsilon > 0 \) and choose \( n_0 \) such that \( 1/n_0 < \varepsilon \). Then from above there is \( N(n_0) \in \mathbb{N} \) such that \( |f_k - f| < 1/n_0 \) on \( A_{n_0} \) for \( k \geq N(n_0) \). Since \( A \subset A_{n_0} \) and \( 1/n_0 < \varepsilon \) then the previous observation implies \( |f_k - f| < \varepsilon \) on \( A \) for \( k \geq N(n_0) \). So \( \{f_n\} \) converges to \( f \) uniformly on \( A \) and \( m(E \setminus A) < \varepsilon/2 \).
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Now we need to find the desired closed set. By Theorem 2.11 there is a closed set \( F \) contained in \( A \) for which \( m(A \setminus F) < \varepsilon/2 \).
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\[ E \setminus F \subset (E \setminus A) \cup (A \setminus F) \]
and
\[ m(E \setminus F) = m(E \setminus A) + m(A \setminus F) < \varepsilon/2 + \varepsilon/2 = \varepsilon. \]
Since \( F \subset A \), then \( \{f_n\} \) converges uniformly on \( F \).
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We now show that \( \{f_n\} \rightarrow f \) uniformly on \( A \). Let \( \varepsilon > 0 \) and choose \( n_0 \) such that \( 1/n_0 < \varepsilon \). Then from above there is \( N(n_0) \in \mathbb{N} \) such that \( |f_k - f| < 1/n_0 \) on \( A_{n_0} \) for \( k \geq N(n_0) \). Since \( A \subset A_{n_0} \) and \( 1/n_0 < \varepsilon \) then the previous observation implies \( |f_k - f| < \varepsilon \) on \( A \) for \( k \geq N(n_0) \). So \( \{f_n\} \) converges to \( f \) uniformly on \( A \) and \( m(E \setminus A) < \varepsilon/2 \).

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Proposition 3.11

**Proposition 3.11.** Let $f$ be a simple function defined on $E$. Then for each $\varepsilon > 0$, there is a continuous function $g$ on $\mathbb{R}$ and a closed set $F$ contained in $E$ for which

$$f = g \text{ on } F \text{ and } m(E \setminus F) < \varepsilon.$$

**Proof.** Let $a_1, a_2, \ldots, a_n$ be the finite number of distinct values taken by $f$ and let the values be taken on the sets $E_1, E_2, \ldots, E_n$ respectively. Since the $a_k$’s are distinct then the $E_k$’s are disjoint.
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$$m(E \setminus F) = m((\bigcup_{k=1}^{n} E_k) \setminus (\bigcup_{k=1}^{n} F_k)) = m(\bigcup_{k=1}^{n} (E_k \setminus F_k))$$

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Proposition 3.11. Let $f$ be a simple function defined on $E$. Then for each $\varepsilon > 0$, there is a continuous function $g$ on $\mathbb{R}$ and a closed set $F$ contained in $E$ for which

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Proof (continued). Define $g$ on $F$ as $g(x) = a_k$ for $x \in F_k$. (The $F_k$'s are disjoint, so $g$ is well-defined.)
Proposition 3.11. Let $f$ be a simple function defined on $E$. Then for each $\varepsilon > 0$, there is a continuous function $g$ on $\mathbb{R}$ and a closed set $F$ contained in $E$ for which

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Proposition 3.11. Let $f$ be a simple function defined on $E$. Then for each $\varepsilon > 0$, there is a continuous function $g$ on $\mathbb{R}$ and a closed set $F$ contained in $E$ for which

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**Proposition 3.11.** Let \( f \) be a simple function defined on \( E \). Then for each \( \varepsilon > 0 \), there is a continuous function \( g \) on \( \mathbb{R} \) and a closed set \( F \) contained in \( E \) for which

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f = g \text{ on } F \text{ and } m(E \setminus F) < \varepsilon.
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**Proof (continued).** Define \( g \) on \( F \) as \( g(x) = a_k \) for \( x \in F_k \). (The \( F_k \)'s are disjoint, so \( g \) is well-defined.) Since the \( F_k \)'s are closed, \( g \) is continuous on \( F \) (for \( x \in F_k \), there is an open interval containing \( x \) which is disjoint from the other \( F_k \)'s, so \( g \) is constant on this open interval intersecting \( F \)). By Problem 3.25, \( g \) can be extended to a function continuous on all of \( \mathbb{R} \). This extension of \( g \) is the desired function. \( \Box \)
Lusin’s Theorem. Let $f$ be a real-valued measurable function on $E$. Then for each $\varepsilon > 0$, there is a continuous function $g$ on $\mathbb{R}$ and a closed set $F$ contained in $E$ for which

$$f = g \text{ on } F \text{ and } m(E \setminus F) < \varepsilon.$$ 

Proof. The case $m(E) = \infty$ is Problem 3.29, so we consider $m(E) < \infty$. 
Lusin’s Theorem

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**Proof.** The case $m(E) = \infty$ is Problem 3.29, so we consider $m(E) < \infty$. By the Simple Approximation Theorem, there is a sequence $\{f_n\}$ of simple functions defined on $E$ that converges to $f$ pointwise on $E$. Let $n \in \mathbb{N}$. 
Lusin’s Theorem

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Lusin’s Theorem. Let $f$ be a real-valued measurable function on $E$. Then for each $\varepsilon > 0$, there is a continuous function $g$ on $\mathbb{R}$ and a closed set $F$ contained in $E$ for which

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Proof. The case $m(E) = \infty$ is Problem 3.29, so we consider $m(E) < \infty$. By the Simple Approximation Theorem, there is a sequence $\{f_n\}$ of simple functions defined on $E$ that converges to $f$ pointwise on $E$. Let $n \in \mathbb{N}$. By Proposition 3.11, with $f$ replaced by $f_n$ and $\varepsilon$ replaced by $\varepsilon/2^{n+1}$, there is a continuous $g_n$ defined on $\mathbb{R}$ and a closed set $F_n$ contained in $E$ for which $f_n = g_n$ on $F_n$ and $m(E \setminus F_n) < \varepsilon/2^{n+1}$. By Egoroff’s Theorem (this is where finite measure is used), there is a closed set $F_0$ contained in $E$ such that $\{f_n\}$ converges to $f$ uniformly on $F_0$ and $m(E \setminus F_0) < \varepsilon/2$. Define $F = \cap_{n=0}^{\infty} F_n$. 


Proof (continued). Then

\[ m(E \setminus F) = m(E \setminus \bigcap_{n=0}^{\infty} F_n) = m(\bigcup_{n=0}^{\infty} (E \setminus F_n)) = m((E \setminus F_0) \cup (\bigcup_{n=1}^{\infty} (E \setminus F_n))) \]

\[ \leq \frac{\varepsilon}{2} + \sum_{n=1}^{\infty} \frac{\varepsilon}{2^{n+1}} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \]

The set \( F \) is closed (since it’s the intersection of closed sets \( F_n \)). Each \( f_n \) is continuous on \( F \) since \( F \subset F_n \) and \( f_n = g_n \) on \( F_n \) and \( g_n \) is continuous on \( \mathbb{R} \). Finally, \( \{f_n\} \) converges to \( f \) uniformly on \( F \) since \( F \subset F_0 \) and \( \{f_n\} \) converges uniformly to \( f \) on \( F_0 \) (that’s how \( F_0 \) was chosen).
Lusin’s Theorem

Proof (continued). Then

\[ m(E \setminus F) = m(E \setminus \bigcap_{n=0}^{\infty} F_n) = m(\bigcup_{n=0}^{\infty} (E \setminus F_n)) = m((E \setminus F_0) \cup (\bigcup_{n=1}^{\infty} (E \setminus F_n))) \]

\[ \leq \frac{\varepsilon}{2} + \sum_{n=1}^{\infty} \frac{\varepsilon}{2^{n+1}} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \]

The set \( F \) is closed (since it’s the intersection of closed sets \( F_n \)). Each \( f_n \) is continuous on \( F \) since \( F \subset F_n \) and \( f_n = g_n \) on \( F_n \) and \( g_n \) is continuous on \( \mathbb{R} \). Finally, \( \{f_n\} \) converges to \( f \) uniformly on \( F \) since \( F \subset F_0 \) and \( \{f_n\} \) converges uniformly to \( f \) on \( F_0 \) (that’s how \( F_0 \) was chosen). However, the uniform limit of continuous functions is continuous, so the restriction of \( f \) to set \( F \) is continuous. By Problem 3.25, there is a continuous function \( g \) defined on all of \( \mathbb{R} \) such that \( g = f \) on \( F \). Function \( g \) is the desired function.
Lusin’s Theorem

Proof (continued). Then

\[ m(E \setminus F) = m(E \setminus \bigcap_{n=0}^{\infty} F_n) = m(\bigcup_{n=0}^{\infty} (E \setminus F_n)) = m((E \setminus F_0) \cup (\bigcup_{n=1}^{\infty} (E \setminus F_n))) \]

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The set \( F \) is closed (since it’s the intersection of closed sets \( F_n \)). Each \( f_n \) is continuous on \( F \) since \( F \subset F_n \) and \( f_n = g_n \) on \( F_n \) and \( g_n \) is continuous on \( \mathbb{R} \). Finally, \( \{f_n\} \) converges to \( f \) uniformly on \( F \) since \( F \subset F_0 \) and \( \{f_n\} \) converges uniformly to \( f \) on \( F_0 \) (that’s how \( F_0 \) was chosen). However, the uniform limit of continuous functions is continuous, so the restriction of \( f \) to set \( F \) is continuous. By Problem 3.25, there is a continuous function \( g \) defined on all of \( \mathbb{R} \) such that \( g = f \) on \( F \). Function \( g \) is the desired function. \( \square \)