Lemma 4.1. Let \( \{E_i\}_{i=1}^n \) be a finite disjoint collection of measurable subsets of a set of finite measure \( E \). For \( 1 \leq i \leq n \), let \( a_i \) be a real number. If \( \varphi = \sum a_i \chi_{E_i} \) on \( E \) then \( \int_E \varphi = \sum a_i m(E_i) \).

**Proof.** Let \( \{\lambda_1, \lambda_2, \ldots, \lambda_m\} \) be the distinct values taken by \( \varphi \). For \( 1 \leq j \leq m \), set \( A_j = \{x \in E \mid \varphi(x) = \lambda_j\} \). Then the canonical representation of \( \varphi \) is \( \varphi = \sum_{j=1}^m \lambda_j \chi_{A_j} \) and so \( \int_E \varphi = \sum_{j=1}^m \lambda_j m(A_j) \). For \( 1 \leq j \leq m \), let \( I_j = \{i \in \{1, 2, \ldots, n\} \mid a_i = \lambda_j \} \). Then \( \{1, 2, \ldots, n\} = \bigcup_{j=1}^m I_j \). By finite additivity, \( m(A_j) = \sum_{i \in I_j} m(E_i) \) for all \( 1 \leq j \leq m \). Therefore

\[
\sum_{i=1}^n a_i m(E_i) = \sum_{j=1}^m \left[ \sum_{i \in I_j} a_i m(E_i) \right] = \sum_{j=1}^m \lambda_j \left[ \sum_{i \in I_j} m(E_i) \right] = \sum_{j=1}^m \lambda_j m(A_j) = \int_E \varphi. \quad \square
\]
Theorem 4.4. Let \( f \) be a bounded measurable function on a set of finite measure \( E \). Then \( f \) is integrable on \( E \).

**Proof.** Let \( n \in \mathbb{N} \). By the Simple Approximation Lemma, for \( \varepsilon = 1/n \) there are simple functions \( \varphi_n \) and \( \psi_n \) on \( E \) for which \( \varphi_n \leq f \leq \psi_n \) on \( E \) and \( 0 \leq \psi_n - \varphi_n \leq 1/n \) on \( E \). By monotonicity and linearity for simple functions (Proposition 4.2) \( 0 \leq \int_E \psi_n - \int_E \varphi_n = \int_E (\psi_n - \varphi_n) \leq \frac{1}{n} m(E) \).

However,

\[
0 \leq \int_E f - \int_E \varphi = \inf \left\{ \int_E \psi \mid \psi \text{ is simple, } \psi \geq f \right\}
- \sup \left\{ \int_E \varphi \mid \varphi \text{ is simple, } \varphi \leq f \right\} \leq \int_E \psi_n - \int_E \varphi_n \leq \frac{1}{n} m(E)
\]

for all \( n \in \mathbb{N} \). Since \( m(E) < \infty \), \( 0 \leq \int_E f - \int_E \varphi \leq 0 \) and so \( \int_E f = \int_E \varphi \) and \( f \) is Lebesgue integrable on \( E \). \( \square \)

---

**Theorem 4.5. Linearity and Monotonicity.**

Let \( f \) and \( g \) be bounded measurable functions on a set of finite measure \( E \). Then for all \( \alpha, \beta \in \mathbb{R} \),

\[
\int_E (\alpha f + \beta g) = \alpha \int_E f + \beta \int_E g.
\]

Moreover, if \( f \leq g \) on \( E \), then \( \int_E f \leq \int_E g \).

**Proof.** By Theorem 4.4, \( \alpha f + \beta g \) is integrable over \( E \). We present the proof in 3 steps.

1. If \( \psi \) is a simple function, then for \( \alpha \neq 0 \), \( \alpha \psi \) is also simple. Let \( \alpha > 0 \). Then

\[
\int_E \alpha f = \inf_{\psi \geq \alpha f} \int_E \psi = \inf_{\psi \geq \alpha f} \int_E \psi = \alpha \inf_{\psi \geq \alpha f} \int_E \psi = \alpha \int_E f.
\]

Proof (continued). Let \( \alpha < 0 \). Then

\[
\int_E \alpha f = \inf_{\psi \leq \alpha f} \int_E \psi = \inf_{\psi \leq \alpha f} \left( \int_E \psi / \alpha \right) = \alpha \sup_{\psi \leq \alpha f} \int_E \varphi = \alpha \int_E f.
\]

Also, if \( \alpha = 0 \) then of course \( 0 = \int_E \alpha f = \int_E f = 0 \).

2. We finish the proof of linearity by considering \( \alpha = \beta = 1 \). Let \( \psi_1 \) and \( \psi_2 \) be simple functions for which \( f \leq \psi_1 \) and \( g \leq \psi_2 \) on \( E \). Then \( \psi_1 + \psi_2 \) is simple and \( f + g \leq \psi_1 + \psi_2 \) on \( E \). By Proposition 4.2 (for simple functions) \( f + g = \inf_{\psi \geq \psi_1 \geq f} \int_E \psi \leq \int_E (\psi_1 + \psi_2) = \int_E \psi_1 + \psi_2 \), or \( \int_E (f + g) \leq \int_E \psi_1 + \int_E \psi_2 \) for all simple \( \psi_1, \psi_2 \) where \( f \leq \psi_1, g \leq \psi_2 \).

Therefore,

\[
\int_E (f + g) \leq \inf_{\psi_2 \geq g} \left( \inf_{\psi_1 \geq f} \left( \int_E \psi_1 + \int_E \psi_2 \right) \right) = \inf_{\psi_2 \geq g} \left( \int_E f + \int_E \psi_2 \right) = \int_E f + \int_E g.
\]

Proof (continued). Now to reverse this inequality. Let \( \varphi_1 \) and \( \varphi_2 \) be simple with \( \varphi_1 \leq f, \varphi_2 \leq g \) on \( E \). Then \( \varphi_1 + \varphi_2 \leq f + g \) on \( E \) is simple. So

\[
\int_E (f + g) = \sup_{\psi \leq f + g} \int_E \psi \geq \int_E (\varphi_1 + \varphi_2) = \int_E \varphi_1 + \int_E \varphi_2.
\]

Therefore

\[
\int_E (f + g) \geq \sup_{\varphi_2 \leq g} \left( \sup_{\varphi_1 \leq f} \left( \int_E \varphi_1 + \int_E \varphi_2 \right) \right) = \sup_{\varphi_2 \leq g} \left( \int_E f + \int_E \varphi_2 \right) = \int_E f + \int_E g.
\]

Therefore, \( \int_E (f + g) = \int_E f + \int_E g \) and linearity follows.
Theorem 4.5. Linearity and Monotonicity.
Let $f$ and $g$ be bounded measurable functions on a set of finite measure $E$. Then for all $\alpha, \beta \in \mathbb{R}$
\[ \int_E (\alpha f + \beta g) = \alpha \int_E f + \beta \int_E g. \]
Moreover, if $f \leq g$ on $E$, then $\int_E f \leq \int_E g$.

Proof (continued). (3) Suppose $f \leq g$ on $E$. By linearity,
\[ \int_E (g - f) = \int_E g - \int_E f. \]
Since $g - f \geq 0$ then $\int_E (g - f) \geq \int_E \phi$ where
\[ \phi \equiv 0 \text{ on } E \] (a simple function less than $g - f$). So $\int_E (g - f) \geq 0$ and
monotonicity follows.

Proposition 4.8. Let $\{f_n\}$ be a sequence of bounded measurable functions
on a set of finite measure on $E$. If $\{f_n\} \to f$ uniformly on $E$, then
\[ \lim_{n \to \infty} \left( \int_E f_n \right) = \int_E \left( \lim_{n \to \infty} f_n \right) = \int_E f. \]

Proof. Since the convergence is uniform and each $f_n$ is bounded, the limit
function $f$ is bounded (there exists $\varepsilon > 0$ and $n \in \mathbb{N}$ such that $|f_n - f| < \varepsilon$
on $E$). Since $f$ is the pointwise limit of a sequence of measurable functions, then $f$ is measurable by Proposition 3.9. Let $\varepsilon > 0$. Choose
$N \in \mathbb{N}$ such that $|f - f_n| < \varepsilon/m(E)$ on $E$ for all $n \geq N$. By the results of
this section:

Proposition 4.8 (continued)

Proposition 4.8. Let $\{f_n\}$ be a sequence of bounded measurable functions
on a set of finite measure on $E$. If $\{f_n\} \to f$ uniformly on $E$, then
\[ \lim_{n \to \infty} \left( \int_E f_n \right) = \int_E \left( \lim_{n \to \infty} f_n \right) = \int_E f. \]

Proof (continued).
\[ \left| \int_E f - \int_E f_n \right| = \left| \int_E (f - f_n) \right| \quad \text{by linearity} \]
\[ \leq \int_E |f - f_n| \quad \text{by Corollary 4.7} \]
\[ \leq \frac{\varepsilon}{m(E)} m(E) = \varepsilon. \]
Therefore $\lim_{n \to \infty} (\int_E f_n) = \int_E f$. \qed

Bounded Convergence Theorem.

Bounded Convergence Theorem.
Let $\{f_n\}$ be a sequence of measurable functions on a set of finite measure $E$. Suppose $\{f_n\}$ is uniformly pointwise bounded on $E$, that is, there is a
number $M \geq 0$ for which $|f_n| \leq M$ on $E$ for all $n$. If $\{f_n\} \to f$ pointwise on
$E$, then
\[ \lim_{n \to \infty} \left( \int_E f_n \right) = \int_E \left( \lim_{n \to \infty} f_n \right) = \int_E f. \]

Proof. First, $f$ is measurable by Proposition 3.9. Since $|f_n(x)| \leq M$ for all $x \in E$ and so $|f(x)| \leq M$ for all $x \in E$. For any measurable $A \subset E$ and
$n \in \mathbb{N}$, we have
\[ \int_E f_n - \int_E f = \int_E (f_n - f) \quad \text{by linearity} \]
\[ = \int_A (f_n - f) + \int_{E \setminus A} f_n + \int_{E \setminus A} (-f) \quad \text{by linearity and Corollary 4.6}. \]
Bounded Convergence Theorem

Proof (continued). So by Corollaries 4.6, 4.7, and monotonicity

\[ \left| \int_E f_n - \int_E f \right| = \left| \int_F (f_n - f) \right| \leq \int_F |f_n - f| = \int_A |f_n - f| + \int_{E \setminus A} |f_n - f| \]

\[ \leq \int_A |f_n - f| + \int_{E \setminus A} 2M = \int_A |f_n - f| + 2M \mu(E \setminus A). \quad (7) \]

Let \( \varepsilon > 0 \). Since \( \mu(E) < \infty \) and \( f \) is real-valued, Egoroff’s Theorem implies that there is a measurable \( A \subset E \) for which \( \{f_n\} \to f \) uniformly on \( A \) and \( \mu(E \setminus A) < \varepsilon/(4M) \). By the uniform convergence on \( A \), there is \( N \in \mathbb{N} \) for which \( |f_n - f| < \frac{\varepsilon}{2\mu(A)} \) on \( A \) for all \( n \geq N \). So for \( n \geq N \), equation (7) implies

\[ \left| \int_E f_n - \int_E f \right| \leq \frac{\varepsilon}{2\mu(A)} \mu(A) + 2M \mu(E \setminus A) < \frac{\varepsilon}{2} + 2M \frac{\varepsilon}{4M} = \varepsilon. \]

So \( \lim_{n \to \infty} (\int_E f_n) = \int_E (\lim_{n \to \infty} f_n) = \int_E f. \quad \square \)