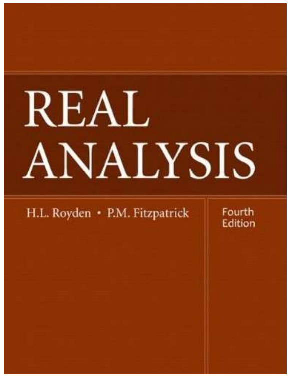
Real Analysis

Chapter 4. Lebesgue Integration

4.2. The Lebesgue Integral of a Bounded Measurable Function over a Set of Finite Measure—Proofs of Theorems



Lemma 4.1

Lemma 4.1. Let $\{E_i\}_{i=1}^n$ be a finite disjoint collection of measurable subsets of a set of finite measure E. For $1 \le i \le n$, let a_i be a real number. If $\varphi = \sum a_i \chi_{E_i}$ on E then $\int_E \varphi = \sum a_i m(E_i)$.

Proof. Let $\{\lambda_1, \lambda_2, \ldots, \lambda_m\}$ be the distinct values taken by φ. For $1 \le j \le m$, set $A_j = \{x \in E \mid \varphi(x) = \lambda_j\}$. Then the canonical representation of φ is $\varphi = \sum_{j=1}^m \lambda_j \chi_{A_j}$ and so $\int_E \varphi = \sum_{j=1}^m \lambda_j m(A_j)$. For $1 \le j \le m$, let $I_j = \{i \in \{1, 2, \ldots, n\} \mid a_i = \lambda_j\}$. Then $\{1, 2, \ldots, n\} = \cup_{j=1}^m I_j$. By finite additivity, $m(A_j) = \sum_{i \in I_j} m(E_i)$ for all $1 \le j \le m$. Therefore

$$\sum_{i=1}^n a_i m(E_i) = \sum_{j=1}^m \left[\sum_{i \in I_j} a_i m(E_i) \right]$$

$$= \sum_{j=1}^m \lambda_j \left[\sum_{i \in I_j} m(E_i) \right] = \sum_{j=1}^m \lambda_j m(A_j) = \int_E \varphi. \quad \square$$

Proposition 4.2

Proposition 4.2. Linearity and Monotonicity of Integration. Let φ and ψ be simple functions defined on a set of finite measure E. Then for any α, β

$$\int_E (\alpha\varphi + \beta\psi) = \alpha \int_E \varphi + \beta \int_E \psi$$

and if $\varphi \le \psi$ on E then $\int_E \varphi \le \int_E \psi$.

Proof. Since both φ and ψ take on a finite number of values on E, we can choose a finite disjoint collection $\{E_i\}_{i=1}^n$ of measurable subsets of E where $\cup E_i = E$ and such that φ and ψ are both constant on each E_i. Let a_i and b_i, respectively, denote the values of φ and ψ on E_i ($1 \le i \le n$). Then representations of φ and ψ (though maybe not the canonical representations since the a_i's may not be distinct and the b_i's may not be distinct) are $\varphi = \sum_{i=1}^n a_i \chi_{E_i}$ and $\psi = \sum_{i=1}^n b_i \chi_{E_i}$. So by Lemma 4.1, $\int_E \varphi = \sum_{i=1}^n a_i m(E_i)$ and $\int_E \psi = \sum_{i=1}^n b_i m(E_i)$.

Proposition 4.2

Proof (continued). The simple function $\alpha\varphi + \beta\psi$ takes on the value $\alpha a_i + \beta b_i$ on E_i and so by Lemma 4.1

$$\int_E (\alpha\varphi + \beta\psi) = \int_E \sum_{i=1}^n (\alpha a_i + \beta b_i)\chi_{E_i} = \sum_{i=1}^n (\alpha a_i + \beta b_i) m(E_i)$$

$$= \alpha \sum_{i=1}^n a_i m(E_i) + \beta \sum_{i=1}^n b_i m(E_i) = \alpha \int_E \varphi + \beta \int_E \psi.$$

To prove monotonicity, let $\varphi \le \psi$ on E and define $\eta = \psi - \varphi$ on E. By the linearity above, $\int_E \psi - \int_E \varphi = \int_E (\psi - \varphi) = \int_E \eta \ge 0$ since η is a nonnegative simple function on E (i.e., $\int_E \eta$ is a sum of nonnegative values times nonnegative measures). $\square$

Theorem 4.3

Theorem 4.3. Let f be a bounded function defined on $[a,b]$. If f is Riemann integrable over $[a,b]$ then it is Lebesgue integrable over $[a,b]$ and the two integrals are equal.

Proof. Recall (see the Riemann-Lebesgue Theorem handout) that upper and lower Riemann integrals are defined in terms of step functions. Since step functions are also simple functions,

$$\underline{R\int_a^b} f(x)\,dx = \sup_{\substack{s \le f \\ s \text{ a step function}}} \left\{\int_a^b s\right\} \le \sup_{\substack{\varphi \le f \\ \varphi \text{ simple}}} \left\{\int_a^b \varphi\right\} = \underline{\int_{[a,b]}} f$$

$$\le \overline{\int_{[a,b]}} f = \inf_{\substack{\psi \ge f \\ \psi \text{ simple}}} \left\{\int_a^b \psi\right\} \le \inf_{\substack{S \ge f \\ S \text{ a step function}}} \left\{\int_a^b S\right\} = R\overline{\int_a^b} f(x)\,dx.$$

If f is Riemann integrable, then the inequalities must be equalities and the Riemann integral equals the Lebesgue integral, as claimed. □

Theorem 4.4

Theorem 4.4. Let f be a bounded measurable function on a set of finite measure E. Then f is integrable on E.

Proof. Let $n \in \mathbb{N}$. By the Simple Approximation Lemma, for $\varepsilon = 1/n$ there are simple functions φ_n and ψ_n on E for which $\varphi_n \le f \le \psi_n$ on E and $0 \le \psi_n - \varphi_n \le 1/n$ on E. By monotonicity and linearity for simple functions (Proposition 4.2) $0 \le \int_E \psi_n - \int_E \varphi_n = \int_E (\psi_n - \varphi_n) \le \frac{1}{n} m(E)$. However,

$$0 \le \overline{\int_E} f - \underline{\int_E} f = \inf\left\{\int_E \psi \mid \psi \text{ is simple}, \psi \ge f\right\}$$

$$-\sup\left\{\int_E \varphi \mid \varphi \text{ is simple}, \varphi \le f\right\} \le \int_E \psi_n - \int_E \varphi_n \le \frac{1}{n} m(E)$$

for all $n \in \mathbb{N}$. Since $m(E) < \infty$, $0 \le \overline{\int_E} f - \underline{\int_E} f \le 0$ and so $\overline{\int_E} f = \underline{\int_E} f$ and f is Lebesgue integrable on E. □

Theorem 4.5

Theorem 4.5. Linearity and Monotonicity.
Let f and g be bounded measurable functions on a set of finite measure E. Then for all $\alpha, \beta \in \mathbb{R}$

$$\int_E (\alpha f + \beta g) = \alpha \int_E f + \beta \int_E g.$$

Moreover, if $f \le g$ on E, then $\int_E f \le \int_E g$.

Proof. By Theorem 4.4, $\alpha f + \beta g$ is integrable over E. We present the proof in 3 steps.
(1) If ψ is a simple function, then for $\alpha \ne 0$, $\alpha\psi$ is also simple. Let $\alpha > 0$. Then

$$\int_E \alpha f = \inf_{\psi \ge \alpha f} \int_E \psi = \inf_{\psi/\alpha \ge f} \int_E \psi = \alpha \inf_{\psi/\alpha \ge f} \int_E \psi/\alpha = \alpha \int_E f.$$

Theorem 4.5 (continued 1)

Proof (continued). Let $\alpha < 0$. Then $\displaystyle\int_E \alpha f = \inf_{\varphi \ge \alpha f} \int_E \varphi = \inf_{\varphi/\alpha \le f} \int_E \varphi =$ $\displaystyle\inf_{\varphi/\alpha \le f} \left(\alpha \int_E \varphi/\alpha\right) = \alpha \sup_{\varphi/\alpha \le f} \int_E \varphi/\alpha = \alpha \int_E f$. Also, if $\alpha = 0$ then of course $0 = \int_E \alpha f = \alpha \int_E f = 0$.

(2) We finish the proof of linearity by considering $\alpha = \beta = 1$. Let ψ_1 and ψ_2 be simple functions for which $f \le \psi_1$ and $g \le \psi_2$ on E. Then $\psi_1 + \psi_2$ is simple and $f + g \le \psi_1 + \psi_2$ on E. By Proposition 4.2 (for simple functions) $\int_E (f+g) = \inf_{\varphi \ge f+g} \int_E \varphi \le \int_E (\psi_1 + \psi_2) = \int_E \psi_1 + \int_E \psi_2$, or $\int_E (f+g) \le \int_E \psi_1 + \int_E \psi_2$ for all simple ψ_1, ψ_2 where $f \le \psi_1, g \le \psi_2$.
Therefore,

$$\int_E (f+g) \le \inf_{\psi_2 \ge g} \left(\inf_{\psi_1 \ge f} \left(\int_E \psi_1 + \int_E \psi_2\right)\right)$$

$$= \inf_{\psi_2 \ge g} \left(\int_E f + \int_E \psi_2\right) = \int_E f + \int_E g.$$

Theorem 4.5 (continued 2)

Proof (continued). Now to reverse this inequality. Let φ_1 and φ_2 be simple with $\varphi_1 \leq f$, $\varphi_2 \leq g$ on E. Then $\varphi_1 + \varphi_2 \leq f + g$ on E is simple. So

$$\int_E (f+g) = \sup_{\varphi \leq f+g} \int_E \varphi \geq \int_E (\varphi_1 + \varphi_2) = \int_E \varphi_1 + \int_E \varphi_2.$$

Therefore

$$\int (f+g) \geq \sup_{\varphi_2 \leq g} \left(\sup_{\varphi_1 \leq f} \left(\int_E \varphi_1 + \int_E \varphi_2 \right) \right)$$

$$= \sup_{\varphi_2 \leq g} \left(\int_E f + \int_E \varphi_2 \right) = \int_E f + \int_E g.$$

Therefore, $\int_E (f+g) = \int_E f + \int_E g$ and linearity follows.

Theorem 4.5 (continued 3)

Theorem 4.5. Linearity and Monotonicity.
Let f and g be bounded measurable functions on a set of finite measure E. Then for all $\alpha, \beta \in \mathbb{R}$

$$\int_E (\alpha f + \beta g) = \alpha \int_E f + \beta \int_E g.$$

Moreover, if $f \leq g$ on E, then $\int_E f \leq \int_E g$.

Proof (continued). (3) Suppose $f \leq g$ on E. By linearity, $\int_E (g-f) = \int_E g - \int_E f$. Since $g - f \geq 0$ then $\int_E (g-f) \geq \int_E \varphi$ where $\varphi \equiv 0$ on E (a simple function less than $g - f$). So $\int_E (g-f) \geq 0$ and monotonicity follows. □

Corollary 4.6

Corollary 4.6. Let f be a bounded measurable function on a set E of finite measure. Suppose A and B are measurable disjoint subsets of E. Then

$$\int_{A \cup B} f = \int_A f + \int_B f.$$

Proof. Both $f \cdot \chi_A$ and $f \cdot \chi_B$ are bounded measurable functions on E. Since A and B are disjoint then $f \cdot \chi_{A \cup B} = f \cdot \chi_A + f \cdot \chi_B$. By Problem 4.10, for any measurable subset E_1 of E we have $\int_{E_1} f = \int_E f \cdot \chi_{E_1}$. So by linearity (Theorem 4.5) we have

$$\int_{A \cup B} f = \int_E f \cdot \chi_{A \cup B} = \int_E (f \cdot \chi_A + f \cdot \chi_B) = \int_E f \cdot \chi_A + \int_E f \cdot \chi_B = \int_A f + \int_B f,$$

as claimed. □

Corollary 4.7

Corollary 4.7. Let f be a bounded measurable function on a set of finite measure E. Then

$$\left| \int_E f \right| \leq \int_E |f|.$$

Proof. The function $|f|$ is measurable by Proposition 3.7. Certainly $|f|$ is bounded. Now $-|f| \leq f \leq |f|$ on E. So by linearity and monotonicity (Theorem 4.5) we have

$$-\int_E |f| \leq \int_E f \leq \int_E |f| \text{ or } \left| \int_E f \right| \leq \int_E |f|,$$

as claimed. □

Proposition 4.8

Proposition 4.8. Let $\{f_n\}$ be a sequence of bounded measurable functions on a set of finite measure on E. If $\{f_n\} \to f$ uniformly on E, then

$$\lim_{n\to\infty}\left(\int_E f_n\right) = \int_E \left(\lim_{n\to\infty} f_n\right) = \int_E f.$$

Proof. Since the convergence is uniform and each f_n is bounded, the limit function f is bounded (there exists $\varepsilon > 0$ and $n \in \mathbb{N}$ such that $|f_n - f| < \varepsilon$ on E). Since f is the pointwise limit of a sequence of measurable functions, then f is measurable by Proposition 3.9. Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $|f - f_n| < \varepsilon/m(E)$ on E for all $n \geq N$. By the results of this section:

Proposition 4.8 (continued)

Proposition 4.8. Let $\{f_n\}$ be a sequence of bounded measurable functions on a set of finite measure on E. If $\{f_n\} \to f$ uniformly on E, then

$$\lim_{n\to\infty}\left(\int_E f_n\right) = \int_E \left(\lim_{n\to\infty} f_n\right) = \int_E f.$$

Proof (continued).

$$\begin{aligned}\left|\int_E f - \int_E f_n\right| &= \left|\int_E (f - f_n)\right| \text{ by linearity (Theorem 4.5)}\\ &\leq \int_E |f - f_n| \text{ by Corollary 4.7}\\ &< \frac{\varepsilon}{m(E)} m(E) = \varepsilon.\end{aligned}$$

Therefore $\lim_{n\to\infty}(\int_E f_n) = \int_E f$. □

Bounded Convergence Theorem

Bounded Convergence Theorem.
Let $\{f_n\}$ be a sequence of measurable functions on a set of finite measure E. Suppose $\{f_n\}$ is uniformly pointwise bounded on E, that is, there is a number $M \geq 0$ for which $|f_n| \leq M$ on E for all n. If $\{f_n\} \to f$ pointwise on E, then

$$\lim_{n\to\infty}\left(\int_E f_n\right) = \int_E \left(\lim_{n\to\infty} f_n\right) = \int_E f.$$

Proof. First, f is measurable by Proposition 3.9. Since $|f_n(x)| \leq M$ for all $x \in E$ and so $|f(x)| \leq M$ for all $x \in E$. For any measurable $A \subset E$ and $n \in \mathbb{N}$, we have

$$\begin{aligned}\int_E f_n - \int_E f &= \int_E (f_n - f) \text{ by linearity (Theorem 4.5)}\\ &= \int_A (f_n - f) + \int_{E\setminus A} (f_n - f) \text{ by Corollary 4.6.}\end{aligned}$$

Bounded Convergence Theorem

Proof (continued). So by Theorem 4.5 and Corollaries 4.6, 4.7,

$$\left|\int_E f_n - \int_E f\right| = \left|\int_E (f_n - f)\right| \leq \int_E |f_n - f| = \int_A |f_n - f| + \int_{E\setminus A} |f_n - f|$$

$$\leq \int_A |f_n - f| + \int_{E\setminus A} 2M = \int_A |f_n - f| + 2Mm(E\setminus A). \quad (7)$$

Let $\varepsilon > 0$. Since $m(E) < \infty$ and f is real-valued, Egoroff's Theorem implies that there is a measurable $A \subset E$ for which $\{f_n\} \to f$ uniformly on A and $m(E \setminus A) < \varepsilon/(4M)$. By the uniform convergence on A, there is $N \in \mathbb{N}$ for which $|f_n - f| < \frac{\varepsilon}{2m(A)}$ on A for all $n \geq N$. So for $n \geq N$, equation (7) implies

$$\left|\int_E f_n - \int_E f\right| < \frac{\varepsilon}{2m(A)} m(A) + 2M\,m(E\setminus A) < \frac{\varepsilon}{2} + 2M\frac{\varepsilon}{4M} = \varepsilon.$$

So $\lim_{n\to\infty}(\int_E f_n) = \int_E(\lim_{n\to\infty} f_n) = \int_E f$. □