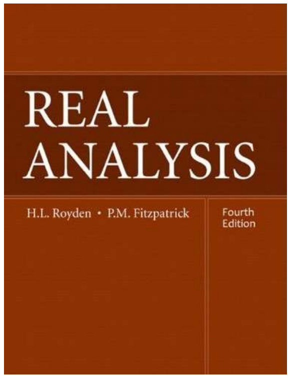
Real Analysis

Chapter 4. Lebesgue Integration

4.3. The Lebesgue Integral of a Measurable Nonnegative Function—Proofs of Theorems



Chebychev's Inequality

Chebychev's Inequality.
Let f be a nonnegative measurable function on E. Then for any $\lambda > 0$,

$$m(\{x \in E \mid f(x) \geq \lambda\}) \leq \frac{1}{\lambda} \int_E f.$$

Proof. Define $E_\lambda = \{x \in E \mid f(x) \geq \lambda\}$.
(1) Suppose $m(E_\lambda) = \infty$. For $n \in \mathbb{N}$, define $E_{\lambda,n} = E_\lambda \cap [-n, n]$ and $\psi_n = \lambda \chi_{E_{\lambda,n}}$. Then ψ_n is a bounded measurable function of finite support (i.e., nonzero on a set of finite measure), $\lambda m(E_{\lambda,n}) = \int_E \psi_n$, and $0 \leq \psi_n \leq f$ on E. By the Continuity of Measure (Theorem 2.15),

$$\infty = \lambda m(E_\lambda) = \lambda \lim_{n\to\infty} m(E_{\lambda,n}) = \lim_{n\to\infty} \int_E \psi_n \leq \int_E f.$$

So $\int_E f = \infty$ and the result holds.

Chebychev's Inequality (continued)

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Let f be a nonnegative measurable function on E. Then for any $\lambda > 0$,

$$m(\{x \in E \mid f(x) \geq \lambda\}) \leq \frac{1}{\lambda} \int_E f.$$

Proof (continued). (2) Suppose $m(E_\lambda) < \infty$. Define $h = \lambda \chi_{E_\lambda}$. Then h is a bounded measurable function of finite support and $0 \leq h \leq f$ on E. So by the definition of integral, $\lambda m(E_\lambda) = \int_E h \leq \int_E f$. The result holds. □

Proposition 4.9

Proposition 4.9. Let f be a nonnegative measurable function on set E. Then $\int_E f = 0$ if and only if $f = 0$ a.e. on E.

Proof. (1) Suppose $\int_E f = 0$. Then by Chebychev's Inequality, for each $n \in \mathbb{N}$, $m(\{x \in E \mid f(x) > 1/n\}) = 0$. By Continuity of Measure (Theorem 2.15)

$$m(\{x \in E \mid f(x) > 0\}) = m(\cup_{n=1}^\infty \{x \in E \mid f(x) > 1/n\})$$

$$= \lim_{n\to\infty} m(\{x \in E \mid f(x) > 1/n\}) = 0.$$

(2) Suppose $f = 0$ a.e. on E. Let φ be a simple function and h a bounded measurable function of finite support for which $0 \leq \varphi \leq h \leq f$ on E. Then $\varphi = 0$ a.e. on E and so $\int_E \varphi = 0$. Since this holds for all such φ, we have that $\int_E h = 0$. Since this holds for all such h, we have that $\int_E f = 0$. □

Theorem 4.10

Theorem 4.10. Linearity and Monotonicity of Integration.
Let f and g be nonnegative measurable functions on E. Then for any $\alpha > 0$ and $\beta > 0$,

$$\int_E (\alpha f + \beta g) = \alpha \int_E f + \beta \int_E g.$$

Moreover, if $f \le g$ on E then $\int_E f \le \int_E g$.

Proof. For $\alpha > 0$, $0 \le h \le f$ on E if and only if $0 \le \alpha h \le \alpha f$ on E. Therefore

$$\begin{aligned}\int_E \alpha f &= \sup\left\{ \int_E \alpha h \,\middle|\, \alpha h \text{ bounded, finite support}, 0 \le \alpha h \le \alpha f \right\} \\ &= \alpha \sup\left\{ \int_E h \,\middle|\, h \text{ bounded, finite support}, 0 \le h \le f \right\} \\ &= \alpha \int_E f.\end{aligned}$$

Theorem 4.10 (continued 1)

Proof (continued). To prove linearity, we only need to consider $\alpha = \beta = 1$. Let h and k be bounded measurable functions of finite support for which $0 \le h \le f$ and $0 \le k \le g$ on E. We have $0 \le h + k \le f + g$ on E, and $h + k$ also is a bounded measurable function of finite support. Thus by linearity of integration (Theorem 4.5),

$$\int_E h + \int_E k = \int_E (h+k) \le \int_E (f+g).$$

Taking suprema over all such h and k gives $\int_E f + \int_E g \le \int_E (f+g)$. Next let $0 \le \ell \le f + g$ on E be a bounded measurable function of finite support. Define $h = \min\{f, \ell\}$ and $k = \ell - h$ on E. For $x \in E$ if $\ell(x) \le f(x)$ then $\ell(x) = h(x) \le f(x)$ and $k(x) = \ell(x) - h(x) = 0 \le g(x)$; if $\ell(x) > f(x)$ then $h(x) = f(x)$ and $k(x) = \ell(x) - h(x) = \ell(x) - f(x) \le g(x)$ (since $\ell(x) \le f(x) + g(x)$). Therefore, $k \le g$ on E.

Theorem 4.10 (continued 2)

Proof (continued). Both h and k are bounded measurable functions of finite support. We have $0 \le h \le f$, $0 \le k \le g$, and $\ell = h + k$ on E. By linearity of the integral (Theorem 4.5), $\int_E \ell = \int_E h + \int_E k \le \int_E f + \int_E g$. Taking a suprema over all such ℓ gives $\int_E (f+g) \le \int_E f + \int_E g$ and linearity follows.

For monotonicity, let h be an arbitrary bounded measurable function of finite support for which $0 \le h \le f$ on E. Since $f \le g$ on E, then $h \le g$ on E and so

$$\int_E h \le \sup\left\{ \int_E h \,\middle|\, h \le g \right\} = \int_E g.$$

Taking a supremum over all such $h \le f$ gives $\int_E f \le \int_E g$. □

Theorem 4.11

Theorem 4.11. Additivity Over Domain of Integration.
Let f be a nonnegative measurable function on E. If A and B are disjoint measurable subsets of E, then

$$\int_{A \cup B} f = \int_A f + \int_B f.$$

In particular, if E_0 is a subset of E of measure zero, then $\int_E f = \int_{E \setminus E_0} f$.

Proof. First, for E_1 a measurable subset of E we have

$$\begin{aligned}\int_{E_1} f &= \sup\{\textstyle\int_{E_1} h \mid h \text{ is bounded, measurable,} \\ &\qquad \text{of finite support, and } 0 \le h \le f \text{ on } E_1\} \\ &= \sup\{\textstyle\int_E h \cdot \chi_{E_1} \mid h \text{ is bounded, measurable,} \\ &\qquad \text{of finite support, and } 0 \le h \cdot \chi_{E_1} \le f \text{ on } E\} \text{ by Problem 4.10}\end{aligned}$$

Theorem 4.11 (continued 1)

Proof (continued). ...

$$\int_{E_1} f = \sup\{\textstyle\int_E h\cdot\chi_{E_1} \mid h \text{ is bounded, measurable, of finite support, and } 0 \le h\cdot\chi_{E_1} \le f \text{ on } E\}$$
$$= \sup\{\textstyle\int_E h\cdot\chi_{E_1} \mid h \text{ is bounded, measurable, of finite support, and } 0 \le h\cdot\chi_{E_1} \le f\cdot\chi_{E_1} \text{ on } E\}$$
$$= \int_E f\cdot\chi_{E_1}.$$

Since A and B are disjoint then $f\cdot\chi_{A\cup B} = f\cdot\chi_A + f\cdot\chi_B$. So by linearity (Theorem 4.10) we have

$$\int_{A\cup B} f = \int_E f\cdot\chi_{A\cup B} = \int_E (f\cdot\chi_A + f\cdot\chi_B) = \int_E f\cdot\chi_A + \int_E f\cdot\chi_B = \int_A f + \int_B f,$$

as claimed.

Theorem 4.11 (continued 2)

Theorem 4.11. Additivity Over Domain of Integration.
Let f be a nonnegative measurable function on E. If A and B are disjoint measurable subsets of E, then

$$\int_{A\cup B} f = \int_A f + \int_B f.$$

In particular, if E_0 is a subset of E of measure zero, then $\int_E f = \int_{E\setminus E_0} f$.

Proof (continued). By Proposition 4.9, $\displaystyle\int_{E_0} f = 0$ since $m(E_0) = 0$. By additivity from above,

$$\int_E f = \int_{E\setminus E_0} f + \int_{E_0} f = \int_{E\setminus E_0} f,$$

as claimed. □

Fatou's Lemma

Fatou's Lemma. Let $\{f_n\}$ be a sequence of nonnegative measurable functions on E. If $\{f_n\} \to f$ pointwise a.e. on E, then

$$\int_E f \le \liminf \left(\int_E f_n\right).$$

Proof. It follows from Theorem 4.11, that the convergence is everywhere WLOG. So f is nonnegative and measurable (by Proposition 3.9). Let h be a bounded measurable function of finite support for which $0 \le h \le f$ on E. Choose $M \ge 0$ for which $|h| \le M$ on E. Define $E_0 = \{x \in E \mid h(x) \ne 0\}$. Then $m(E_0) < \infty$ since h is of finite support. Let $n \in \mathbb{N}$. Define h_n on E as $h_n = \min\{h, f_n\}$. Then h_n is measurable (by Proposition 3.8), $0 \le h_n \le M$ on E_0 and $h_n = 0$ on $E \setminus E_0$ (since $h = 0$ there). Also, for each $x \in E$, since $h(x) \le f(x)$ and $\{f_n(x)\} \to f(x)$, then $\{h_n(x)\} \to h(x)$.

Fatou's Lemma (continued)

Proof (continued). Applying the Bounded Convergence Theorem to $\{h_n\}$,

$$\lim_{n\to\infty}\left(\int_E h_n\right) = \lim_{n\to\infty}\left(\int_{E_0} h_n\right) = \int_{E_0}\left(\lim_{n\to\infty} h_n\right) = \int_{E_0} h = \int_E h.$$

Since $h_n \le f_n$ on E and h_n is bounded and of finite support, by the definition of $\int_E f_n$, $\int_E h_n \le \int_E f_n$. Therefore

$$\int_E h = \lim_{n\to\infty}\left(\int_E h_n\right) \le \liminf\left(\int_E f_n\right).$$

Since h is an arbitrary bounded function of finite support and $h \le f$, then

$$\int_E f = \sup\left\{\int_E h \mid h \text{ bounded, finite support, } 0 \le h \le f\right\} \le \liminf\left(\int_E f_n\right).$$

□

Monotone Convergence Theorem

Monotone Convergence Theorem. Let $\{f_n\}$ be an increasing sequence of nonnegative measurable functions on E. If $\{f_n\} \to f$ pointwise a.e. on E, then

$$\lim_{n\to\infty}\left(\int_E f_n\right) = \int_E \left(\lim_{n\to\infty} f_n\right) = \int_E f.$$

Proof. Since the sequence $\{f_n\}$ is increasing, then $f_n \le f$ almost everywhere on E. So by the monotonicity of integration (Theorem 4.10), $\int_E f_n \le \int_E f$. Therefore $\limsup\left(\int_E f_n\right) \le \int_E f$. By Fatou's Lemma, $\int_E f \le \liminf\left(\int_E f_n\right)$. Since $\limsup \ge \liminf$, it follows that

$$\lim_{n\to\infty}\left(\int_E f_n\right) = \int_E f.$$

□

Corollary 4.12

Corollary 4.12. Let $\{u_n\}$ be a sequence of nonnegative measurable functions on E. If $f = \sum_{n=1}^{\infty} u_n$ pointwise a.e. on E, then

$$\int_E f = \sum_{n=1}^{\infty}\left(\int_E u_n\right).$$

Proof. Since each u_n is nonnegative, then $\sum_{n=1}^{k} u_n$ is an increasing sequence of nonnegative measurable functions. So

$$\begin{aligned}\int_E f &= \int_E \sum_{n=1}^{\infty} u_n = \int_E \lim_{k\to\infty}\sum_{n=1}^{k} u_n \\ &= \lim_{k\to\infty}\int_E \sum_{n=1}^{k} u_n \text{ by the Monotone Convergence Theorem}\end{aligned}$$

Corollary 4.12 (continued)

Corollary 4.12. Let $\{u_n\}$ be a sequence of nonnegative measurable functions on E. If $f = \sum_{n=1}^{\infty} u_n$ pointwise a.e. on E, then

$$\int_E f = \sum_{n=1}^{\infty}\left(\int_E u_n\right).$$

Proof (continued). Since each u_n is nonnegative, then $\sum_{n=1}^{k} u_k$ is an increasing sequence of nonnegative measurable functions. So

$$\begin{aligned}\int_E f &= \lim_{k\to\infty}\int_E \sum_{n=1}^{k} u_n = \lim_{k\to\infty}\sum_{n=1}^{k}\int_E u_n \text{ by linearity (Theorem 4.10)} \\ &= \sum_{n=1}^{\infty}\int_E u_n,\end{aligned}$$

as claimed. □

Proposition 4.13

Proposition 4.13. Let nonnegative f be integrable over E. Then f is finite a.e. on E.

Proof. Let $n \in \mathbb{N}$. By monotonicity of measure and Chebychev's Inequality,

$$m(\{x\in E \mid f(x)=\infty\}) \le m(\{x \in E \mid f(x) \ge n\}) \le \frac{1}{n}\int_E f.$$

Since $\int_E f < \infty$ and this holds for all $n \in \mathbb{N}$, it must be that $m(\{x \in E \mid f(x) = \infty\}) = 0$. □

Beppo Levi's Lemma

Beppo Levi's Lemma. Let $\{f_n\}$ be an increasing sequence of nonnegative measurable functions on E. If the sequence $\{\int_E f_n\}$ is bounded, then $\{f_n\}$ converges pointwise on E to a measurable function f that is finite a.e. on E and

$$\lim_{n\to\infty}\left(\int_E f_n\right) = \int_E f < \infty.$$

Proof. Every monotone sequence of extended real numbers converges to an extended real number. So $\{f_n\}$ converges pointwise on E and is measurable (by Proposition 3.8). By the Monotone Convergence Theorem, $\left\{\int_E f_n\right\} \to \int_E f$. Since $\{\int_E f_n\}$ is bounded, its limit is finite and so $\int_E f < \infty$. By Proposition 4.13, f is finite a.e. on E. □