Chapter 4. Lebesgue Integration
4.3. The Lebesgue Integral of a Measurable Nonnegative Function—Proofs of Theorems
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Chebychev’s Inequality

Chebychev’s Inequality. Let $f$ be a nonnegative measurable function on $E$. Then for any $\lambda > 0$,

$$m\left(\{x \in E \mid f(x) \geq \lambda\}\right) \leq \frac{1}{\lambda} \int_E f.$$

Proof. Define $E_\lambda = \{x \in E \mid f(x) \geq \lambda\}$. 

Real Analysis
Chebychev’s Inequality

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(1) Suppose $m(E_\lambda) = \infty$. For $n \in \mathbb{N}$, define $E_{\lambda,n} = E_\lambda \cap [-n, n]$ and $\psi_n = \lambda \chi_{E_{\lambda,n}}$. 
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Let $f$ be a nonnegative measurable function on $E$. Then for any $\lambda > 0$, \[ m(\{x \in E \mid f(x) \geq \lambda\}) \leq \frac{1}{\lambda} \int_E f. \]

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1. Suppose $m(E_\lambda) = \infty$. For $n \in \mathbb{N}$, define $E_{\lambda,n} = E_\lambda \cap [-n, n]$ and $\psi_n = \lambda \chi_{E_{\lambda,n}}$. Then $\psi_n$ is a bounded measurable function of “finite support” (i.e., nonzero on a set of finite measure), and $\lambda m(E_{\lambda,n}) = \int_E \psi_n$ and $0 \leq \psi_n \leq f$ on $E$. 

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November 17, 2016
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Let $f$ be a nonnegative measurable function on $E$. Then for any $\lambda > 0$,

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$$\infty = \lambda m(E_\lambda) = \lambda \lim_{n \to \infty} m(E_{\lambda,n}) = \lim_{n \to \infty} \int_E \psi_n \leq \int_E f.$$

So $\int_E f = \infty$ and the result holds.
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Let $f$ be a nonnegative measurable function on $E$. Then for any $\lambda > 0$,

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Proof (continued). (2) Suppose $m(E_\lambda) < \infty$. Define $h = \lambda \chi_{E_\lambda}$. Then $h$ is a bounded measurable function of finite support and $0 \leq h \leq f$ on $E$. So by the definition of integral, $\lambda m(E_\lambda) = \int_E h \leq \int_E f$. The result holds. $\Box$
Chebychev’s Inequality (continued)

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Proposition 4.9

Let \( f \) be a nonnegative measurable function on set \( E \). Then \( \int_E f = 0 \) if and only if \( f = 0 \) a.e. on \( E \).

Proof. (1) Suppose \( \int_E f = 0 \). Then by Chebychev’s Inequality, for each \( n \in \mathbb{N} \), \( m(\{x \in E \mid f(x) > 1/n\}) = 0 \).
Proposition 4.9. Let $f$ be a nonnegative measurable function on set $E$. Then $\int_E f = 0$ if and only if $f = 0$ a.e. on $E$.

Proof. (1) Suppose $\int_E f = 0$. Then by Chebychev’s Inequality, for each $n \in \mathbb{N}$, $m(\{x \in E \mid f(x) > 1/n\}) = 0$. By Continuity of Measure (Theorem 2.15)

$$m(\{x \in E \mid f(x) > 0\}) = m(\bigcup_{n=1}^{\infty} \{x \in E \mid f(x) > 1/n\})$$

$$= \lim_{n \to \infty} m(\{x \in E \mid f(x) > 1/n\}) = 0.$$
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(2) Suppose $f = 0$ a.e. on $E$. Let $\varphi$ be a simple function and $h$ a bounded measurable function of finite support for which $0 \leq \varphi \leq h \leq f$ on $E$. Then $\varphi = 0$ a.e. on $E$ and so $\int_E \varphi = 0$. 
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Theorem 4.10

Theorem 4.10. Linearity and Monotonicity of Integration.
Let \( f \) and \( g \) be nonnegative measurable functions on \( E \). Then for any \( \alpha > 0 \) and \( \beta > 0 \),

\[
\int_E (\alpha f + \beta g) = \alpha \int_E f + \beta \int_E g.
\]

Moreover, if \( f \leq g \) on \( E \) then \( \int_E f \leq \int_E g \).

Proof. For \( \alpha > 0 \), \( 0 \leq h \leq f \) on \( E \) if and only if \( 0 \leq \alpha h \leq \alpha f \) on \( E \). Therefore

\[
\int_E \alpha f = \sup \left\{ \int_E \alpha h \ \bigg| \ \text{\( \alpha h \) bounded, finite support, } 0 \leq \alpha h \leq \alpha f \right\}
\]

\[
= \alpha \sup \left\{ \int_E h \ \bigg| \ h \text{ bounded, finite support, } 0 \leq h \leq f \right\}
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Theorem 4.10. Linearity and Monotonicity of Integration.

Let $f$ and $g$ be nonnegative measurable functions on $E$. Then for any $\alpha > 0$ and $\beta > 0$,

$$\int_E (\alpha f + \beta g) = \alpha \int_E f + \beta \int_E g.$$ 

Moreover, if $f \leq g$ on $E$ then $\int_E f \leq \int_E g$.

**Proof.** For $\alpha > 0$, $0 \leq h \leq f$ on $E$ if and only if $0 \leq \alpha h \leq \alpha f$ on $E$. Therefore

$$\int_E \alpha f = \sup \left\{ \int_E \alpha h \mid \alpha h \text{ bounded, finite support, } 0 \leq \alpha h \leq \alpha f \right\}$$

$$= \alpha \sup \left\{ \int_E h \mid h \text{ bounded, finite support, } 0 \leq h \leq f \right\}$$

$$= \alpha \int_E f.$$
Theorem 4.10 (continued 1)

**Proof (continued).** To prove linearity, we only need to consider \( \alpha = \beta = 1 \). Let \( h \) and \( k \) be bounded measurable functions of finite support for which \( 0 \leq h \leq f \) and \( 0 \leq k \leq g \) on \( E \). We have \( 0 \leq h + k \leq f + g \) on \( E \), and \( h + k \) also is a bounded measurable function of finite support. Thus by linearity of integration (Theorem 4.5),

\[
\int_E h + \int_E k = \int_E (h + k) \leq \int_E (f + g).
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Proof (continued). To prove linearity, we only need to consider $\alpha = \beta = 1$. Let $h$ and $k$ be bounded measurable functions of finite support for which $0 \leq h \leq f$ and $0 \leq k \leq g$ on $E$. We have $0 \leq h + k \leq f + g$ on $E$, and $h + k$ also is a bounded measurable function of finite support. Thus by linearity of integration (Theorem 4.5),

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Taking suprema over all such $h$ and $k$ gives $\int_E f + \int_E g \leq \int_E (f + g)$. 
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Taking suprema over all such \( h \) and \( k \) gives \( \int_E f + \int_E g \leq \int_E (f + g) \). Next let \( 0 \leq \ell \leq f + g \) on \( E \) be a bounded measurable function of finite support. Define \( h = \min\{f, \ell\} \) and \( k = \ell - h \) on \( E \).
Proof (continued). To prove linearity, we only need to consider $\alpha = \beta = 1$. Let $h$ and $k$ be bounded measurable functions of finite support for which $0 \leq h \leq f$ and $0 \leq k \leq g$ on $E$. We have $0 \leq h + k \leq f + g$ on $E$, and $h + k$ also is a bounded measurable function of finite support. Thus by linearity of integration (Theorem 4.5),

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Proof (continued). To prove linearity, we only need to consider $\alpha = \beta = 1$. Let $h$ and $k$ be bounded measurable functions of finite support for which $0 \leq h \leq f$ and $0 \leq k \leq g$ on $E$. We have $0 \leq h + k \leq f + g$ on $E$, and $h + k$ also is a bounded measurable function of finite support. Thus by linearity of integration (Theorem 4.5),

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Theorem 4.10 (continued 2)

**Proof (continued).** Both $h$ and $k$ are bounded measurable functions of finite support. We have $0 \leq h \leq f$, $0 \leq k \leq g$, and $\ell = h + k$ on $E$. By linearity of the integral (Theorem 4.5), $\int_E \ell = \int_E h + \int_E k \leq \int_E f + \int_E g$. Taking a suprema over all such $\ell$ gives $\int_E (f + g) \leq \int_E f + \int_E g$ and linearity follows.
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For monotonicity, let $h$ be an arbitrary bounded measurable function of finite support for which $0 \leq h \leq f$ on $E$. Since $f \leq g$ on $E$, then $h \leq g$ on $E$ and so

$$\int_E h \leq \sup \left\{ \int_E h \bigg| h \leq g \right\} = \int_E g.$$
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Taking a supremum over all such \( h \leq f \) gives \( \int_E f \leq \int_E g \). \qed
Fatou’s Lemma

**Fatou’s Lemma.** Let \( \{f_n\} \) be a sequence of nonnegative measurable functions on \( E \). If \( \{f_n\} \to f \) pointwise a.e. on \( E \), then

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\int_E f \leq \liminf \left( \int_E f_n \right).
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**Proof.** It follows from Theorem 4.11, that the convergence is everywhere WLOG. So \( f \) is nonnegative and measurable (by Proposition 3.9).
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Fatou’s Lemma (continued)

Proof (continued). Applying the Bounded Convergence Theorem to \( \{h_n\} \),

\[
\lim_{n \to \infty} \left( \int_E h_n \right) = \lim_{n \to \infty} \left( \int_{E_0} h_n \right) = \int_{E_0} \left( \lim_{n \to \infty} h_n \right) = \int_{E_0} h = \int_E h.
\]

Since \( h_n \leq f_n \) on \( E \) and \( h_n \) is bounded and of finite support, by the definition of \( \int_E f_n \), \( \int_E h_n \leq \int_E f_n \). Therefore

\[
\int_E h = \lim_{n \to \infty} \left( \int_E h_n \right) \leq \lim \inf \left( \int_E f_n \right).
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\int_E h = \lim_{n \to \infty} \left( \int_E h_n \right) \leq \lim \inf \left( \int_E f_n \right).
\]

Since \( h \) is an arbitrary bounded function of finite support and \( h \leq f \), then

\[
\int_E f \leq \lim \inf \left( \int_E f_n \right).
\]
Fatou’s Lemma (continued)

**Proof (continued).** Applying the Bounded Convergence Theorem to \( \{h_n\} \),

\[
\lim_{n \to \infty} \left( \int_E h_n \right) = \lim_{n \to \infty} \left( \int_{E_0} h_n \right) = \int_{E_0} \left( \lim_{n \to \infty} h_n \right) = \int_{E_0} h = \int_E h.
\]

Since \( h_n \leq f_n \) on \( E \) and \( h_n \) is bounded and of finite support, by the definition of \( \int_E f_n \), \( \int_E h_n \leq \int_E f_n \). Therefore

\[
\int_E h = \lim_{n \to \infty} \left( \int_E h_n \right) \leq \lim \inf \left( \int_E f_n \right).
\]

Since \( h \) is an arbitrary bounded function of finite support and \( h \leq f \), then

\[
\int_E f \leq \lim \inf \left( \int_E f_n \right).
\]
Monotone Convergence Theorem. Let \( \{f_n\} \) be an increasing sequence of nonnegative measurable functions on \( E \). If \( \{f_n\} \to f \) pointwise a.e. on \( E \), then

\[
\lim_{n \to \infty} \left( \int_{E} f_n \right) = \int_{E} \left( \lim_{n \to \infty} f_n \right) = \int_{E} f.
\]

Proof. Since the sequence \( \{f_n\} \) is increasing, then \( f_n \leq f \) almost everywhere on \( E \). So by the monotonicity of integration (Theorem 4.10),

\[
\int_{E} f_n \leq \int_{E} f.
\]

Therefore \( \lim \sup (\int_{E} f_n) \leq \int_{E} f \).
Monotone Convergence Theorem. Let \( \{f_n\} \) be an increasing sequence of nonnegative measurable functions on \( E \). If \( \{f_n\} \to f \) pointwise a.e. on \( E \), then

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Proof. Since the sequence \( \{f_n\} \) is increasing, then \( f_n \leq f \) almost everywhere on \( E \). So by the monotonicity of integration (Theorem 4.10),

\[
\int_E f_n \leq \int_E f.
\]

Therefore

\[
\limsup_{n \to \infty} (\int_E f_n) \leq \int_E f.
\]

By Fatou’s Lemma,

\[
\int_E f \leq \liminf (\int_E f_n).
\]

Since \( \limsup \geq \liminf \), it follows that

\[
\lim_{n \to \infty} \left( \int_E f_n \right) = \int_E f.
\]
Monotone Convergence Theorem. Let \( \{f_n\} \) be an increasing sequence of nonnegative measurable functions on \( E \). If \( \{f_n\} \to f \) pointwise a.e. on \( E \), then

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\lim_{n \to \infty} \left( \int_E f_n \right) = \int_E \left( \lim_{n \to \infty} f_n \right) = \int_E f.
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Proof. Since the sequence \( \{f_n\} \) is increasing, then \( f_n \leq f \) almost everywhere on \( E \). So by the monotonicity of integration (Theorem 4.10), \( \int_E f_n \leq \int_E f \). Therefore \( \limsup (\int_E f_n) \leq \int_E f \). By Fatou’s Lemma, \( \int_E f \leq \liminf (\int_E f_n) \). Since \( \limsup \geq \liminf \), it follows that

\[
\lim_{n \to \infty} \left( \int_E f_n \right) = \int_E f.
\]
Proposition 4.13. Let nonnegative $f$ be integrable over $E$. Then $f$ is finite a.e. on $E$.

Proof. Let $n \in \mathbb{N}$. By monotonicity of measure and Chebychev’s Inequality,

$$m(\{x \in E | f(x) = \infty\}) \leq m(\{x \in E | f(x) \geq n\}) \leq \frac{1}{n} \int_E f.$$
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Beppo Levi’s Lemma. Let \( \{f_n\} \) be an increasing sequence of nonnegative measurable functions on \( E \). If the sequence \( \{\int_E f_n\} \) is bounded, then \( \{f_n\} \) converges pointwise on \( E \) to a measurable function \( f \) that is finite a.e. on \( E \) and
\[
\lim_{n \to \infty} \left( \int_E f_n \right) = \int_E f < \infty.
\]

Proof. Every monotone sequence of extended real numbers converges to an extended real number. So \( \{f_n\} \) converges pointwise on \( E \) and is measurable (by Proposition 3.8).
Beppo Levi’s Lemma. Let \( \{ f_n \} \) be an increasing sequence of nonnegative measurable functions on \( E \). If the sequence \( \{ \int_E f_n \} \) is bounded, then \( \{ f_n \} \) converges pointwise on \( E \) to a measurable function \( f \) that is finite a.e. on \( E \) and

\[
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\]

Proof. Every monotone sequence of extended real numbers converges to an extended real number. So \( \{ f_n \} \) converges pointwise on \( E \) and is measurable (by Proposition 3.8). By the Monotone Convergence Theorem, \( \{ \int_E f_n \} \to \int_E f \). Since \( \{ \int_E f_n \} \) is bounded, its limit is finite and so \( \int_E f < \infty \).
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