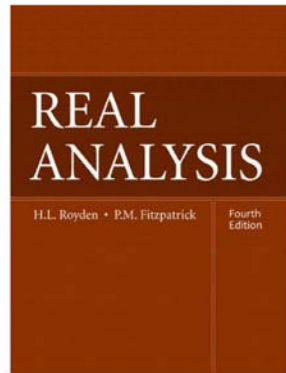


Real Analysis

Chapter 4. Lebesgue Integration

4.4. The General Lebesgue Integral—Proofs of Theorems



Proposition 4.14

Proposition 4.14. Let f be a measurable function on E . Then f^+ and f^- are integrable over E if and only if $|f|$ is integrable over E .

Proof. Suppose f^+ and f^- are integrable over E ; that is, $\int_E f^+ < \infty$ and $\int_E f^- < \infty$. Now f^+ and f^- are nonnegative, so by the linearity of integration of nonnegative functions (Theorem 4.10),

$$\int_E |f| = \int_E (f^+ + f^-) = \int_E f^+ + \int_E f^- < \infty$$

and $|f|$ is integrable over E .

Suppose $|f|$ is integrable over E ; that is, $\int_E |f| < \infty$. Now $0 \leq f^+ \leq |f|$ and $0 \leq f^- \leq |f|$ on E , so by monotonicity of integration for nonnegative functions (Theorem 4.10),

$$0 \leq \int_E f^+ \leq \int_E |f| < \infty \text{ and } 0 \leq \int_E f^- \leq \int_E |f| < \infty.$$

Then f^+ and f^- are integrable over E . □

Proposition 4.15

Proposition 4.15. Let f be integrable over E . Then f is finite a.e. on E and $\int_E f = \int_{E \setminus E_0} f$ if $E_0 \subset E$ and $m(E_0) = 0$.

Proof. By Proposition 4.13, $|f|$ is finite a.e. on E . So f is finite a.e. on E , as claimed. Next,

$$\begin{aligned} \int_E f &= \int_E f^+ - \int_E f^- \text{ by definition of } \int_E f \\ &= \int_{E \setminus E_0} f^+ - \int_{E \setminus E_0} f^- \text{ by additivity of integration (Theorem 4.11)} \\ &= \int_{E \setminus E_0} f \text{ by definition of } \int_{E \setminus E_0} f. \end{aligned}$$

The claim holds. □

Proposition 4.16. The Integral Comparison Test

Proposition 4.16. The Integral Comparison Test.

Let f be a measurable function on E . Suppose there is a nonnegative function g that is integrable over E and $|f| \leq g$ on E . Then f is integrable over E and $|\int_E f| \leq \int_E |f|$.

Proof. By monotonicity of integration for nonnegative functions (Theorem 4.10), $|f|$ is integrable and so by Proposition 4.14, f is integrable. Therefore

$$\begin{aligned} \left| \int_E f \right| &= \left| \int_E f^+ - \int_E f^- \right| \text{ by definition} \\ &\leq \left(\int_E f^+ \right) + \left(\int_E f^- \right) \text{ by the triangle inequality on } \mathbb{R} \\ &= \int_E (f^+ + f^-) \text{ by linearity for nonnegative functions (Thm. 4.10)} \\ &= \int_E |f|. \end{aligned}$$

Theorem 4.17. Linearity and Monotonicity of Integration

Theorem 4.17. Linearity and Monotonicity of Integration.

Let the functions f and g be integrable over E . Then for any α and β , the function $\alpha f + \beta g$ is integrable over E and

$$\int_E (\alpha f + \beta g) = \alpha \int_E f + \beta \int_E g.$$

Also, if $f \leq g$ on E , then $\int_E f \leq \int_E g$.

Proof. If $\alpha > 0$, then $(\alpha f)^+ = \alpha f^+$ and $(\alpha f)^- = \alpha f^-$. If $\alpha < 0$, then $(\alpha f)^+ = -\alpha f^-$ and $(\alpha f)^- = -\alpha f^+$. So for $\alpha > 0$,

$$\begin{aligned} \int_E \alpha f &= \int_E ((\alpha f)^+ - (\alpha f)^-) = \int_E (\alpha f^+) - \int_E (\alpha f^-) \text{ by definition} \\ &= \int_E \alpha f^+ - \int_E \alpha f^- = \alpha \int_E f^+ - \alpha \int_E f^- \text{ by linearity (Thm. 4.10)} \\ &= \alpha \left(\int_E f^+ - \int_E f^- \right) = \alpha \int_E f \text{ by definition.} \end{aligned}$$

()

Theorem 4.17 (continued 1)

Proof (continued). For $\alpha < 0$,

$$\begin{aligned} \int_E \alpha f &= \int_E ((\alpha f)^+ - (\alpha f)^-) \\ &= \int_E (\alpha f^+) - \int_E (\alpha f^-) \text{ by definition} \\ &= \int_E (-\alpha f^-) - \int_E (-\alpha f^+) \\ &= (-\alpha) \int_E f^- - (-\alpha) \int_E f^+ \text{ by linearity (Theorem 4.10)} \\ &= \alpha \left(-\int_E f^- + \int_E f^+ \right) = \alpha \int_E f \text{ by definition.} \end{aligned}$$

To complete the proof of linearity, we need to show that

$\int_E (f + g) = (\int_E f) + (\int_E g)$. By linearity for nonnegative functions (Theorem 4.10), $|f| + |g|$ is integrable over E .

()

Theorem 4.17 (continued 2)

Proof (continued). Pointwise, $|f + g| \leq |f| + |g|$ on E , then by the Integral Comparison Test (Proposition 4.16),

$|\int_E (f + g)| \leq \int_E |f + g| \leq \int_E |f| + \int_E |g|$ and $f + g$ is integrable over E . By Proposition 4.15, $f + g$ is finite a.e. on E and f and g are finite a.e. on E . So WLOG, f and g are finite on E . Now

$(f + g)^+ - (f + g)^- = f + g = (f^+ - f^-) + (g^+ - g^-)$ on E and so $(f + g)^+ + f^- + g^- = (f + g)^- + f^+ + g^+$ on E . So by linearity of integration (Theorem 4.10),

$$\int_E (f + g)^+ + \int_E f^- + \int_E g^- = \int_E (f + g)^- + \int_E f^+ + \int_E g^+.$$

Since f , g , and $f + g$ are integrable over E , each of these integrals is finite. Rearranging, we have

$$\int_E f^+ - \int_E f^- + \int_E g^+ - \int_E g^- = \int_E (f + g)^+ - \int_E (f + g)^-$$

or $\int_E f + \int_E g = \int_E (f + g)$. This establishes linearity.

()

Theorem 4.17. Linearity and Monotonicity of Integration (continued 3)

Proof (continued). To establish monotonicity, we again observe that WLOG f and g can be taken as finite on E . Define $h = g - f$ on E (and so h is nonnegative WLOG on E ; we are avoiding $\infty - \infty$ here). By linearity of integration for integrable functions (part (a)) and monotonicity for nonnegative functions (Theorem 4.10),

$$\begin{aligned} \int_E g - \int_E f &= \int_E (g - f) \text{ by part(a)} \\ &= \int_E h \\ &\geq 0 \text{ by Theorem 4.10,} \end{aligned}$$

or $\int_E f \leq \int_E g$. □

()

Corollary 4.18. Additivity Over Domains of Integration

Corollary 4.18. Additivity Over Domains of Integration.

Let f be integrable over E . Assume A and B are disjoint measurable subsets of E . Then

$$\int_{A \cup B} f = \int_A f + \int_B f.$$

Proof. First, we have (pointwise) that $|f \cdot \chi_A| \leq |f|$ and $|f \cdot \chi_B| \leq |f|$ on E . By the Integral Comparison Test (Proposition 4.16), the measurable function $f \cdot \chi_A$ and $f \cdot \chi_B$ are integrable over E . Since A and B are disjoint, then $f \cdot \chi_{A \cup B} = f \cdot \chi_A + f \cdot \chi_B$ on E . By Exercise 4.28, for any $C \in \mathcal{M}$, $C \subset E$, $\int_C f = \int_E (f \cdot \chi_C)$. The result follows from linearity for integrable functions (Theorem 4.17). \square

Theorem. The Lebesgue Dominated Convergence Theorem

Theorem. The Lebesgue Dominated Convergence Theorem.

Let $\{f_n\}$ be a sequence of measurable functions on E . Suppose there is a function g that is integrable over E and dominates $\{f_n\}$ in the sense that $|f_n| \leq g$ on E for all n . If $\{f_n\} \rightarrow f$ pointwise a.e. on E , then f is integrable over E and

$$\lim_{n \rightarrow \infty} \left(\int_E f_n \right) = \int_E \left(\lim_{n \rightarrow \infty} f_n \right) = \int_E f.$$

Proof. Since $|f_n| \leq g$ on E and $\{f_n\} \rightarrow f$ a.e. on E , then $|f| \leq g$ a.e. on E . Since g is integrable over E , by the Integral Comparison Test (Proposition 4.16), f and each f_n are integrable over E . By Proposition 4.15, WLOG f and each f_n are finite on E . Therefore, functions $g - f$ and $g - f_n$ for each n are “properly defined” (i.e., there is no $\infty - \infty$ here), nonnegative and measurable. Moreover, the sequence $\{g - f_n\}$ converges pointwise a.e. on E to $g - f$. By Fatou’s Lemma, $\int_E (g - f) \leq \liminf \left(\int_E (g - f_n) \right)$.

Theorem. The Lebesgue Dominated Convergence Theorem (continued)

Theorem. The Lebesgue Dominated Convergence Theorem.

Let $\{f_n\}$ be a sequence of measurable functions on E . Suppose there is a function g that is integrable over E and dominates $\{f_n\}$ in the sense that $|f_n| \leq g$ on E for all n . If $\{f_n\} \rightarrow f$ pointwise a.e. on E , then f is integrable over E and

$$\lim_{n \rightarrow \infty} \left(\int_E f_n \right) = \int_E \left(\lim_{n \rightarrow \infty} f_n \right) = \int_E f.$$

Proof (continued). By linearity of integration (Theorem 4.17),

$$\int_E g - \int_E f = \int_E (g - f) \leq \liminf \left(\int_E (g - f_n) \right) = \int_E g - \limsup \left(\int_E f_n \right).$$

That is, $\limsup \left(\int_E f_n \right) \leq \int_E f$. Similarly, considering $\{g + f_n\}$, we have $\int_E f \leq \liminf \left(\int_E f_n \right)$ and the result follows. \square