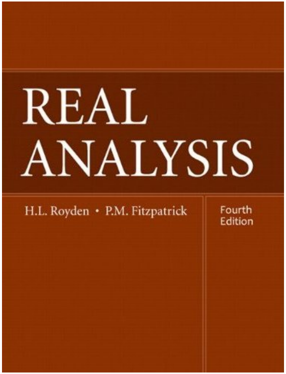
Real Analysis

Chapter 4. Lebesgue Integration

4.4. The General Lebesgue Integral—Proofs of Theorems



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Proof. Suppose f^+ and f^- are integrable over E; that is, $\int_E f^+ < \infty$ and $\int_E f^- < \infty$. Now f^+ and f^- are nonnegative, so by the linearity of integration of nonnegative functions (Theorem 4.10),

$$\int_E |f| = \int_E (f^+ + f^-) = \int_E f^+ + \int_E f^- < \infty$$

and $|f|$ is integrable over E.

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Suppose $|f|$ is integrable over E; that is, $\int_E |f| < \infty$. Now $0 \leq f^+ \leq |f|$ and $0 \leq f^- \leq |f|$ on E, so by monotonicity of integration for nonnegative functions (Theorem 4.10),

$$0 \leq \int_E f^+ \leq \int_E |f| < \infty \text{ and } 0 \leq \int_E f^- \leq \int_E |f| < \infty.$$

Then f^+ and f^- are integrable over E. $\square$

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Then f^+ and f^- are integrable over E. □

Proposition 4.15

Proposition 4.15. Let f be integrable over E. Then f is finite a.e. on E and $\int_E f = \int_{E \setminus E_0} f$ if $E_0 \subset E$ and $m(E_0) = 0$.

Proof. By Proposition 4.13, $|f|$ is finite a.e. on E. So f is finite a.e. on E, as claimed.

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Proof. By Proposition 4.13, $|f|$ is finite a.e. on E. So f is finite a.e. on E, as claimed. Next,

$$\begin{aligned}\int_E f &= \int_E f^+ - \int_E f^- \text{ by definition of } \int_E f \\ &= \int_{E\setminus E_0} f^+ - \int_{E\setminus E_0} f^- \text{ by additivity of integration (Theorem 4.11)} \\ &= \int_{E\setminus E_0} f \text{ by definition of } \int_{E\setminus E_0} f.\end{aligned}$$

The claim holds. □

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Proposition 4.16. The Integral Comparison Test

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Let f be a measurable function on E. Suppose there is a nonnegative function g that is integrable over E and $|f| \leq g$ on E. Then f is integrable over E and $\left|\int_E f\right| \leq \int_E |f|$.

Proof. By monotonicity of integration for nonnegative functions (Theorem 4.10), $|f|$ is integrable and so by Proposition 4.14, f is integrable.

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Proof. By monotonicity of integration for nonnegative functions (Theorem 4.10), $|f|$ is integrable and so by Proposition 4.14, f is integrable. Therefore

$$\begin{aligned} \left|\int_E f\right| &= \left|\int_E f^+ - \int_E f^-\right| \text{ by definition} \\ &\leq \left(\int_E f^+\right) + \left(\int_E f^-\right) \text{ by the triangle inequality on } \mathbb{R} \\ &= \int_E (f^+ + f^-) \text{ by linearity for nonnegative functions (Thm. 4.10)} \\ &= \int_E |f|. \end{aligned}$$

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$$\begin{aligned}
\left|\int_E f\right| &= \left|\int_E f^+ - \int_E f^-\right| \text{ by definition} \\
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&= \int_E (f^+ + f^-) \text{ by linearity for nonnegative functions (Thm. 4.10)} \\
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Let the functions f and g be integrable over E. Then for any α and β, the function $\alpha f + \beta g$ is integrable over E and

$$\int_E (\alpha f + \beta g) = \alpha \int_E f + \beta \int_E g.$$

Also, if $f \leq g$ on E, then $\int_E f \leq \int_E g$.

Proof. If $\alpha > 0$, then $(\alpha f)^+ = \alpha f^+$ and $(\alpha f)^- = \alpha f^-$. If $\alpha < 0$, then $(\alpha f)^+ = -\alpha f^-$ and $(\alpha f)^- = -\alpha f^+$.

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$$\begin{aligned}
\int_E \alpha f &= \int_E ((\alpha f)^+ - (\alpha f)^-) = \int_E (\alpha f)^+ - \int_E (\alpha f)^- \text{ by definition} \\
&= \int_E \alpha f^+ - \int_E \alpha f^- = \alpha \int_E f^+ - \alpha \int_E f^- \text{ by linearity (Thm. 4.10)} \\
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Theorem 4.17 (continued 1)

Proof (continued). For $\alpha < 0$,

$$\begin{aligned}
\int_E \alpha f &= \int_E ((\alpha f)^+ - (\alpha f)^-) \\
&= \int_E (\alpha f)^+ - \int_E (\alpha f)^- \text{ by definition} \\
&= \int_E (-\alpha f^-) - \int_E (-\alpha f^+) \\
&= (-\alpha) \int_E f^- - (-\alpha) \int_E f^+ \text{ by linearity (Theorem 4.10)} \\
&= \alpha \left(-\int_E f^- + \int_E f^+ \right) = \alpha \int_E f \text{ by definition.}
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To complete the proof of linearity, we need to show that $\int_E (f+g) = (\int_E f) + (\int_E g)$. By linearity for nonnegative functions (Theorem 4.10), $|f| + |g|$ is integrable over E.

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Theorem 4.17 (continued 2)

Proof (continued). Pointwise, $|f+g| \leq |f| + |g|$ on E, then by the Integral Comparison Test (Proposition 4.16), $|\int_E (f+g)| \leq \int_E |f+g| \leq \int_E |f| + \int_E |g|$ and $f+g$ is integrable over E. By Proposition 4.15, $f+g$ is finite a.e. on E and f and g are finite a.e. on E. So WLOG, f and g are finite on E. Now $(f+g)^+ - (f+g)^- = f + g = (f^+ - f^-) + (g^+ - g^-)$ on E and so $(f+g)^+ + f^- + g^- = (f+g)^- + f^+ + g^+$ on E. So by linearity of integration (Theorem 4.10),

$$\int_E (f+g)^+ + \int_E f^- + \int_E g^- = \int_E (f+g)^- + \int_E f^+ + \int_E g^+.$$

Since f, g, and $f+g$ are integrable over E, each of these integrals is finite.

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$$\int_E f^+ - \int_E f^- + \int_E g^+ - \int_E g^- = \int_E (f+g)^+ - \int_E (f+g)^-$$

or $\int_E f + \int_E g = \int_E (f+g)$. This establishes linearity.

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Theorem 4.17. Linearity and Monotonicity of Integration (continued 3)

Proof (continued). To establish monotonicity, we again observe that WLOG f and g can be taken as finite on E. Define $h = g - f$ on E (and so h is nonnegative WLOG on E; we are avoiding $\infty - \infty$ here). By linearity of integration for integrable functions (part (a)) and monotonicity for nonnegative functions (Theorem 4.10),

$$\begin{aligned} \int_E g - \int_E f &= \int_E (g - f) \text{ by part(a)} \\ &= \int_E h \\ &\geq 0 \text{ by Theorem 4.10,} \end{aligned}$$

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Corollary 4.18. Additivity Over Domains of Integration

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Let f be integrable over E. Assume A and B are disjoint measurable subsets of E. Then

$$\int_{A \cup B} f = \int_A f + \int_B f.$$

Proof. First, we have (pointwise) that $|f \cdot \chi_A| \leq |f|$ and $|f \cdot \chi_B| \leq |f|$ on E.

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Theorem. The Lebesgue Dominated Convergence Theorem

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Let $\{f_n\}$ be a sequence of measurable functions on E. Suppose there is a function g that is integrable over E and dominates $\{f_n\}$ in the sense that $|f_n| \leq g$ on E for all n. If $\{f_n\} \to f$ pointwise a.e. on E, then f is integrable over E and

$$\lim_{n\to\infty}\left(\int_E f_n\right) = \int_E \left(\lim_{n\to\infty} f_n\right) = \int_E f.$$

Proof. Since $|f_n| \leq g$ on E and $\{f_n\} \to f$ a.e. on E, then $|f| \leq g$ a.e. on E. Since g is integrable over E, by the Integral Comparison Test (Proposition 4.16), f and each f_n are integrable over E.

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$$\lim_{n\to\infty}\left(\int_E f_n\right) = \int_E \left(\lim_{n\to\infty} f_n\right) = \int_E f.$$

Proof (continued). By linearity of integration (Theorem 4.17),

$$\int_E g - \int_E f = \int_E (g-f) \leq \liminf\left(\int_E (g-f_n)\right) = \int_E g - \limsup\left(\int_E f_n\right).$$

That is, $\limsup\left(\int_E f_n\right) \leq \int_E f$. Similarly, considering $\{g+f_n\}$, we have $\int_E f \leq \liminf\left(\int_E f_n\right)$ and the result follows. □

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