Real Analysis
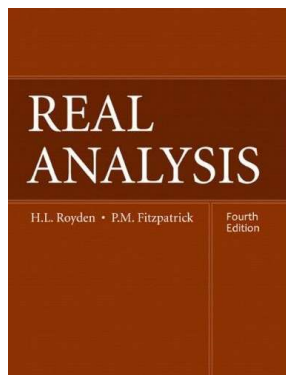
Chapter 4. Lebesgue Integration

4.5. Countable Additivity and Continuity of Integration—Proofs of Theorems

REAL ANALYSIS

H.L. Royden • P.M. Fitzpatrick   Fourth Edition

Theorem 4.20. The Countable Additivity of Integration

Theorem 4.20. The Countable Additivity of Integration.
Let f be integrable over E and $\{E_n\}_{n=1}^{\infty}$ a disjoint collection of measurable subsets of E whose union is E. Then

$$\int_E f = \sum_{n=1}^{\infty} \left(\int_{E_n} f \right).$$

Proof. Let $n \in \mathbb{N}$ and define $f_n = f \cdot \chi_n$ where χ_n is the characteristic function on $\cup_{k=1}^{n} E_k$. The f_n are measurable and integrable by Problem 4.28. Also $|f_n| \leq |f|$ on E (in fact, $f_n(x) = f(x)$ for $x \in \cup_{k=1}^{n} E_k$ and $f_n(x) = 0$ for $x \notin \cup_{k=1}^{n} E_k$). So by the Lebesgue Dominated Convergence Theorem, $\int_E f = \int_E (\lim_{n \to \infty} f_n) = \lim_{n \to \infty} (\int_E f_n)$. Since the E_n's are disjoint, by additivity (Corollary 4.18) we have for each n that $\int_E f_n = \sum_{k=1}^{n} (\int_{E_k} f_n) = \sum_{k=1}^{n} (\int_{E_k} f)$ since $f = f_n$ on E_k. Therefore

$$\int_E f = \lim_{n \to \infty} \left(\int_E f_n \right) = \lim_{n \to \infty} \left(\sum_{k=1}^{n} \int_{E_k} f \right) = \sum_{k=1}^{\infty} \left(\int_{E_k} f \right). \quad \square$$