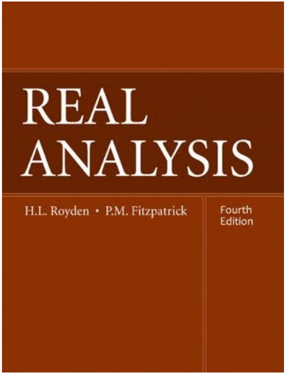
Real Analysis

Chapter 4. Lebesgue Integration

4.5. Countable Additivity and Continuity of Integration—Proofs of Theorems



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Let f be integrable over E and $\{E_n\}_{n=1}^{\infty}$ a disjoint collection of measurable subsets of E whose union is E. Then

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Proof. Let $n \in \mathbb{N}$ and define $f_n = f \cdot \chi_n$ where χ_n is the characteristic function on $\cup_{k=1}^{n} E_k$. The f_n are measurable and integrable by Problem 4.28.

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Proof. Let $n \in \mathbb{N}$ and define $f_n = f \cdot \chi_n$ where χ_n is the characteristic function on $\cup_{k=1}^n E_k$. The f_n are measurable and integrable by Problem 4.28. Also $|f_n| \le |f|$ on E (in fact, $f_n(x) = f(x)$ for $x \in \cup_{k=1}^n E_k$ and $f_n(x) = 0$ for $x \notin \cup_{k=1}^n E_k$). So by the Lebesgue Dominated Convergence Theorem, $\int_E f = \int_E (\lim_{n\to\infty} f_n) = \lim_{n\to\infty}(\int_E f_n)$.

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