Lemma 4.22. Let $E$ be a set of finite measure and $\delta > 0$. Then $E$ is the disjoint union of a finite collection of sets, each of which has measure less than $\delta$.

Proof. By the continuity of measure (Theorem 2.15(ii)),

$$0 = (\emptyset)m = (((\infty, \infty) \cup (-\infty, -\infty) \cup [E, -\infty) \cup [\infty, E)) \cup m(\emptyset)$$

Now partition $[E, -\infty)$ into subintervals of length less than $\delta$.

The definition of limit implies there exists $N \in \mathbb{N}$ for which $m(E \cap [\infty, N)) = \lim_{n \to \infty} m(E \cap [n, \infty))$

by the definition of limit. There exists a disjoint finite collection of sets, $\{A_1, A_2, \ldots, A_N\}$, such that $m(E \cap [\infty, N)) = \sum_{i=1}^{N} m(A_i)$.

Theorem 4.6. Uniform Integrability. The Vitali Convergence Theorem—Proofs of

Chapter 4, Lebesgue Integration

Real Analysis

Proposition 4.2.3. Let $f$ be a measurable function on $E$. If $f$ is Lebesgue integrable over $E$ then there is a $g \geq 0$ for which

$$\int_E f \, d\mu \geq 0$$

Conversely, for $m(E) > \infty$, if for each $\epsilon > 0$, there is a $g \geq 0$ for which

$$\int_E f \, d\mu \geq \epsilon$$

Proposition 4.23 (continued I).
Proposition 4.25. Assume $\mathcal{E}$ has finite measure. Let the sequence of functions $(f_n)$ be uniformly integrable over $\mathcal{E}$. Therefore, by Definition (b), for $\epsilon > 0$ there is $\delta > 0$ such that $m(\mathcal{E}) > r$ for which $(\mathcal{E})$ holds. For $\epsilon > 0$ in $\delta$ the corresponding $\delta$-neighborhood of $\mathcal{E}$ with $(\mathcal{E}) $ holds. For $\epsilon > 0$ in $\delta > 0$ there is a measurable set $\mathcal{E}$ and support for all $\epsilon < 0$, the set $\mathcal{E}$ is uniformly integrable.

Let $\mathcal{E}$ be a probability space and $\mathcal{F}$ be the free sub-$\sigma$-algebra of $\mathcal{E}$. Then, by Definition (c), for $\epsilon > 0$ there is $\delta > 0$ such that $m(\mathcal{E}) > r$ for which $(\mathcal{E})$ holds. For $\epsilon > 0$ in $\delta$ the corresponding $\delta$-neighborhood of $\mathcal{E}$ with $(\mathcal{E}) $ holds. For $\epsilon > 0$ in $\delta > 0$ there is a measurable set $\mathcal{E}$ and support for all $\epsilon < 0$, the set $\mathcal{E}$ is uniformly integrable.

Let $E$ be of finite measure. Suppose $\{f_n\}$ is a sequence of measurable functions that converges pointwise a.e., on $E$ to $f$. Then each $f_n$ is uniformly integrable on $E$.

Proof.

Suppose $\{f_n\}$ is not uniformly integrable on $E$. Then there exists $\varepsilon > 0$ and a measurable set $G \subseteq E$ with $m(G) > 0$ such that $\lim_{N \to \infty} \int_G |f_n| \, dm = \infty$ for all $n \geq N$. Therefore, $f_n$ is not integrable on $G$ for all $n \geq N$. This is a contradiction.

Suppose, by contradiction, that $f_n$ is uniformly integrable on $E$. Then for any $\varepsilon > 0$, there exists $\delta > 0$ such that $\int_E |f_n| \, dm < \varepsilon$ for all $n$ and $\varepsilon < \delta$. Since $f_n$ is uniformly integrable, there exists $N \in \mathbb{N}$ such that $m(E_n) < \delta$ for all $n > N$. Therefore, $\int_G |f_n| \, dm < \varepsilon$ for all $n > N$. This is a contradiction.

Therefore, $f_n$ is uniformly integrable on $E$. Q.E.D.

Theorem 4.27.

Let $E$ be of finite measure. Suppose $\{f_n\}$ is a sequence of measurable functions that converges pointwise a.e., on $E$ to $f$. Then $f$ is measurable.

Proof.

Since $f_n$ is measurable and $E$ is measurable, $f_n \cdot 1_E$ is measurable for all $n$. Therefore, $\lim_{n \to \infty} f_n \cdot 1_E$ is measurable. By the monotone convergence theorem, $\lim_{n \to \infty} f_n \cdot 1_E = \lim_{n \to \infty} f_n \cdot 1_E$. Therefore, $f$ is measurable.

The Vitali convergence theorem (continued)

The Vitali convergence theorem (continued)

The Vitali convergence theorem (continued)