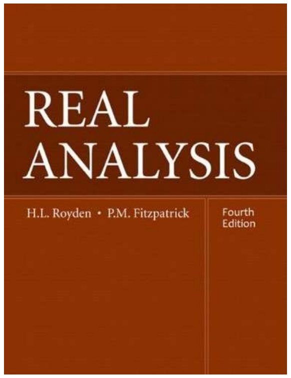
Real Analysis

Chapter 4. Lebesgue Integration

4.6. Uniform Integrability: The Vitali Convergence Theorem—Proofs of Theorems



Lemma 4.22

Lemma 4.22. Let E be a set of finite measure and $\delta > 0$. Then E is the disjoint union of a finite collection of sets, each of which has measure less than δ.

Proof. By the Continuity of Measure (Theorem 2.15(ii)),

$$\lim_{n\to\infty} m(E \setminus [-n,n]) = \lim_{n\to\infty} m(E \cap ((-\infty,-n) \cup (n,\infty))) = m(\varnothing) = 0,$$

since $E \cap ((-\infty,-n) \cup (n,\infty))$ is a descending collection of sets. So by the definition of limit, there exists $n_0 \in \mathbb{N}$ for which $m(E \setminus [-n_0,n_0]) < \delta$. Now partition $[-n_0,n_0]$ into subintervals of length less than δ, say producing intervals $I_1, I_2, \ldots, I_N$. Then E is the disjoint union of $E \setminus [-n_0,n_0] = E \cap ((-\infty,-n) \cap (n,\infty))$, $E \cap I_1$, $E \cap I_2$, $\ldots$, $E \cap I_N$, and each is of measure less than δ. □

Proposition 4.23

Proposition 4.23. Let f be a measurable function on E. If f is integrable over E, then for each $\varepsilon > 0$, there is $\delta > 0$ for which:

$$\text{if } A \subseteq E \text{ is measurable and } m(A) < \delta \text{ then } \int_A |f| < \varepsilon. \tag{26}$$

Conversely, for $m(E) < \infty$, if for each $\varepsilon > 0$, there is a $\delta > 0$ for which (26) holds, then f is integrable over E.

Proof. WLOG, we assume $f \geq 0$ on E (otherwise, we use an $\varepsilon/2$ argument on the positive and negative parts of f). Assume f is integrable over E and let $\varepsilon > 0$. Then, by the definition of the integral of a nonnegative integrable function, there is a bounded function of finite support (that is, nonzero on a set of finite measure), say f_ε, for which $0 \leq f_\varepsilon \leq f$ on E and $0 \leq \int_E f - \int_E f_\varepsilon < \varepsilon/2$ (recall that $\int_E f$ is defined as a supremum over such f_ε).

Proposition 4.23 (continued 1)

Proof (continued). If $A \subseteq E$ is measurable then

$$\begin{aligned}
\int_A f - \int_A f_\varepsilon &= \int_A (f - f_\varepsilon) \text{ by linearity, Theorem 4.17} \\
&\leq \int_E (f - f_\varepsilon) \text{ by additivity over } A \cup (E \setminus A) \text{ since } f - f_\varepsilon \geq 0 \\
&= \int_E f - \int_E f_\varepsilon \text{ by linearity} \\
&< \frac{\varepsilon}{2} \text{ by above.}
\end{aligned}$$

Since f_ε is bounded, choose $M > 0$ for which $0 \leq f_\varepsilon < M$ on E. So for $A \subseteq E$ measurable,

$$\begin{aligned}
\int_A f &< \left(\int_A f_\varepsilon\right) + \frac{\varepsilon}{2} \text{ by the above inequality} \\
&\leq Mm(A) + \frac{\varepsilon}{2}.
\end{aligned}$$

Proposition 4.23 (continued 2)

Proof (continued). Define $\delta = \varepsilon/(2M)$, and this inequality implies for $m(A) < \delta = \varepsilon/(2M)$ that $\int_A f < M(\varepsilon/(2M)) + \varepsilon/2 = \varepsilon$.

Conversely, suppose $m(E) < \infty$ and suppose that for all $\varepsilon > 0$, there is a $\delta > 0$ for which (26) holds. For $\varepsilon = 1$, let $\delta_0 > 0$ be the corresponding δ. Since $m(E) < \infty$, by Lemma 4.22, we may express E as the disjoint union of a finite collection of measurable subsets $\{E_k\}_{k=1}^N$, each of measure less than δ_0. Therefore (by additivity, Theorem 4.11, and since $\varepsilon = 1$),

$$\int_E |f| = \int_{\cup E_k} |f| = \sum_{k=1}^N \left(\int_{E_k} |f| \right) < \sum_{k=1}^N (1) = N.$$

So f is integrable over E. □

Proposition 4.24

Proposition 4.24. Let $\{f_k\}_{k=1}^n$ be a finite collection of functions, each of which is integrable over E. Then $\{f_k\}_{k=1}^n$ is uniformly integrable.

Proof. Let $\varepsilon > 0$. Since each f_k is integrable over E, by Proposition 4.23 we have for each $1 \le k \le n$ that there exists $\delta_k > 0$ such that if $A \subseteq E$ is measurable and $m(A) < \delta_k$ then $\int_A |f_k| < \varepsilon$. Let $\delta = \min\{\delta_1, \delta_2, \ldots, \delta_n\}$. Then with $A \subseteq E$ where $m(A) < \delta$ we have $\int_A |f_k| < \varepsilon$ for each $1 \le k \le n$ and so the family $\mathcal{F} = \{f_k\}_{k=1}^n$ is uniformly integrable, as claimed. □

Proposition 4.25

Proposition 4.25. Assume E has finite measure. Let the sequence of functions $\{f_k\}_{k=1}^\infty$ be uniformly integrable over E. If $\{f_n\} \to f$ pointwise a.e. on E, then f is integrable over E.

Proof. First, by Proposition 3.9, f is measurable. Let $\varepsilon = 1$ and let $\delta_0 > 0$ be the corresponding $\delta > 0$ given by the definition of uniform integrability. Since E has finite measure, by Lemma 4.22, we may express E as a disjoint union of a finite collection of measurable sets $\{E_k\}_{k=1}^N$ such that $m(E_k) < \delta_0$ for $1 \le k \le N$. Then (by additivity, Theorem 4.11, and since $\varepsilon = 1$)

$$\int_E |f_n| = \int_{\cup E_k} |f_n| = \sum_{k=1}^N \left(\int_{E_k} |f_n| \right) < \sum_{k=1}^N (1) = N.$$

So, by Fatou's Lemma, $\int_E |f| \le \liminf \int_E |f_n| \le N$. So $|f|$ is integrable over E and so (by definition) f is integrable over E. □

The Vitali Convergence Theorem

The Vitali Convergence Theorem.
Let E be of finite measure. Suppose sequence $\{f_n\}$ is uniformly integrable over E. If $\{f_n\} \to f$ pointwise a.e. on E, then f is integrable over E and

$$\lim_{n\to\infty} \left(\int_E f_n \right) = \int_E \left(\lim_{n\to\infty} f_n \right) = \int_E f.$$

Proof. By Proposition 4.25, f is integrable over E. So by Proposition 4.15, f is finite a.e. on E. Also by Proposition 4.15, we can "excise" a subset of E of measure 0 (on which the pointwise convergence does not hold) and assume WLOG that f is real valued (as opposed to extended real valued) and that the convergence is pointwise on all of E.

The Vitali Convergence Theorem (continued 1)

Proof (continued). For measurable $A \subseteq E$ and $n \in \mathbb{N}$

$$\begin{aligned} \left| \int_E f_n - \int_E f \right| &= \left| \int_E (f_n - f) \right| \text{ by linearity, Theorem 4.17} \\ &\leq \int_E |f_n - f| \text{ by Proposition 4.16,} \\ &\quad \text{the Integral Comparison Test} \\ &= \int_{E \setminus A} |f_n - f| + \int_A |f_n - f| \text{ by additivity, Theorem 4.11} \\ &\leq \int_{E \setminus A} |f_n - f| + \int_A (|f_n| + |f|) \text{ by the Triangle Inequality} \\ &\quad \text{and Monotonicity (Theorem 4.10)} \\ &= \int_{E \setminus A} |f_n - f| + \int_A |f_n| + \int_A |f| \text{ by linearity.} \qquad (29) \end{aligned}$$

The Vitali Convergence Theorem (continued 2)

Proof (continued). Let $\varepsilon > 0$. Since $\{f_n\}$ is uniformly integrable, there is $\delta > 0$ such that

$$\int_A |f_n| < \varepsilon/3 \text{ for all } n \in \mathbb{N}$$
$$\text{and any measurable } A \subseteq E \text{ with } m(A) < \delta. \qquad (*)$$

By Fatou's Lemma

$$\int_A |f| \leq \liminf \int_A |f_n| \leq \frac{\varepsilon}{3}. \qquad (**)$$

Since f is real-valued and E has finite measure, then by Egoroff's Theorem, there is $E_0 \subseteq E$ with $m(E_0) < \delta$ and $\{f_n\} \to f$ uniformly on $E \setminus E_0$. So there is $N \in \mathbb{N}$ such that $|f_n - f| < \varepsilon/(3m(E))$ on $E \setminus E_0$ for all $n \geq N$.

The Vitali Convergence Theorem (continued 3)

Proof (continued). With $A = E_0$ in (29) we have for $n \geq N$,

$$\begin{aligned} \left| \int_E f_n - \int_E f \right| &\leq \int_{E \setminus E_0} |f_n - f| + \int_{E_0} |f_n| + \int_{E_0} |f| \\ &< \left(\frac{\varepsilon}{3m(E)} \right) m(E \setminus E_0) + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \text{ by } (*) \text{ and } (**) \\ &\leq \varepsilon. \end{aligned}$$

So $\int_E f_n \to \int_E f$. □

Theorem 4.26

Theorem 4.26. Let E be of finite measure. Suppose $\{h_n\}$ is a sequence of nonnegative integrable functions that converges pointwise a.e. on E to $h \equiv 0$. Then

$$\lim_{n \to \infty} \left(\int_E h_n \right) = 0 \text{ if and only if } \{h_n\} \text{ is uniformly integrable over } E.$$

Proof. If $\{h_n\}$ is uniformly integrable, then by the Vitali Convergence Theorem, $\lim_{n\to\infty}(\int_E h_n) = \int_E (\lim_{n\to\infty} h_n) = \int_E 0 = 0$.

Conversely, suppose $\lim_{n\to\infty}(\int_E h_n) = 0$ and let $\varepsilon > 0$. Then there is $N \in \mathbb{N}$ for which $\int_E h_n < \varepsilon$ for $n \geq N$. Since $h_n \geq 0$ on E, then if $A \subseteq E$ is measurable and $n \geq N$ then by monotonicity (Theorem 4.10),

$$\int_A h_n \leq \int_E h_n < \varepsilon \text{ for } n \geq N. \qquad (30)$$

Theorem 4.26 (continued)

Theorem 4.26. Let E be of finite measure. Suppose $\{h_n\}$ is a sequence of nonnegative integrable functions that converges pointwise a.e. on E to $h \equiv 0$. Then

$$\lim_{n\to\infty}\left(\int_E h_n\right) = 0 \text{ if and only if } \{h_n\} \text{ is uniformly integrable over } E.$$

Proof (continued). By Proposition 4.24, since each h_n is integrable, $\{h_n\}_{n=1}^{N-1}$ is uniformly integrable over E. Let $\delta > 0$ correspond to $\varepsilon > 0$ for this set in the definition of uniformly integrable. Then, trivially by (30), this $\delta > 0$ also works for all h_n with $n \geq N$. Therefore $\{h_n\}_{n=1}^{\infty}$ is uniformly integrable. □