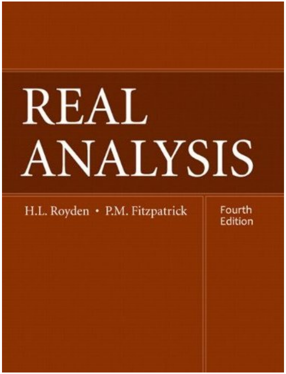
Real Analysis

Chapter 4. Lebesgue Integration

4.6. Uniform Integrability: The Vitali Convergence Theorem—Proofs of Theorems



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Proof. By the Continuity of Measure (Theorem 2.15(ii)),

$$\lim_{n\to\infty} m(E \setminus [-n,n]) = \lim_{n\to\infty} m(E \cap ((-\infty,-n) \cup (n,\infty))) = m(\varnothing) = 0,$$

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Proposition 4.23

Proposition 4.23. Let f be a measurable function on E. If f is integrable over E, then for each $\varepsilon > 0$, there is $\delta > 0$ for which:

$$\text{if } A \subseteq E \text{ is measurable and } m(A) < \delta \text{ then } \int_A |f| < \varepsilon. \tag{26}$$

Conversely, for $m(E) < \infty$, if for each $\varepsilon > 0$, there is a $\delta > 0$ for which (26) holds, then f is integrable over E.

Proof. WLOG, we assume $f \geq 0$ on E (otherwise, we use an $\varepsilon/2$ argument on the positive and negative parts of f). Assume f is integrable over E and let $\varepsilon > 0$.

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Proof (continued). If $A \subseteq E$ is measurable then

$$\begin{aligned} \int_A f - \int_A f_\varepsilon &= \int_A (f - f_\varepsilon) \text{ by linearity, Theorem 4.17} \\ &\leq \int_E (f - f_\varepsilon) \text{ by additivity over } A \mathbin{\dot\cup} (E \setminus A) \text{ since } f - f_\varepsilon \geq 0 \\ &= \int_E f - \int_E f_\varepsilon \text{ by linearity} \\ &< \frac{\varepsilon}{2} \text{ by above.} \end{aligned}$$

Since f_ε is bounded, choose $M > 0$ for which $0 \leq f_\varepsilon < M$ on E. So for $A \subseteq E$ measurable,

$$\begin{aligned} \int_A f &< \left(\int_A f_\varepsilon \right) + \frac{\varepsilon}{2} \text{ by the above inequality} \\ &\leq Mm(A) + \frac{\varepsilon}{2}. \end{aligned}$$

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Proof (continued). Define $\delta = \varepsilon/(2M)$, and this inequality implies for $m(A) < \delta = \varepsilon/(2M)$ that $\int_A f < M(\varepsilon/(2M)) + \varepsilon/2 = \varepsilon$.

Conversely, suppose $m(E) < \infty$ and suppose that for all $\varepsilon > 0$, there is a $\delta > 0$ for which (26) holds. For $\varepsilon = 1$, let $\delta_0 > 0$ be the corresponding δ. Since $m(E) < \infty$, by Lemma 4.22, we may express E as the disjoint union of a finite collection of measurable subsets $\{E_k\}_{k=1}^N$, each of measure less than δ_0.

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$$\int_E |f| = \int_{\cup E_k} |f| = \sum_{k=1}^N \left(\int_{E_k} |f| \right) < \sum_{k=1}^N (1) = N.$$

So f is integrable over E. □

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Proposition 4.24. Let $\{f_k\}_{k=1}^n$ be a finite collection of functions, each of which is integrable over E. Then $\{f_k\}_{k=1}^n$ is uniformly integrable.

Proof. Let $\varepsilon > 0$. Since each f_k is integrable over E, by Proposition 4.23 we have for each $1 \leq k \leq n$ that there exists $\delta_k > 0$ such that if $A \subseteq E$ is measurable and $m(A) < \delta_k$ then $\int_A |f_k| < \varepsilon$.

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Proof. First, by Proposition 3.9, f is measurable.

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Let E be of finite measure. Suppose sequence $\{f_n\}$ is uniformly integrable over E. If $\{f_n\} \to f$ pointwise a.e. on E, then f is integrable over E and

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The Vitali Convergence Theorem (continued 1)

Proof (continued). For measurable $A \subseteq E$ and $n \in \mathbb{N}$

$$\begin{aligned}
\left|\int_E f_n - \int_E f\right| &= \left|\int_E (f_n - f)\right| \text{ by linearity, Theorem 4.17} \\
&\leq \int_E |f_n - f| \text{ by Proposition 4.16,} \\
&\quad \text{the Integral Comparison Test} \\
&= \int_{E\setminus A} |f_n - f| + \int_A |f_n - f| \text{ by additivity, Theorem 4.11} \\
&\leq \int_{E\setminus A} |f_n - f| + \int_A (|f_n| + |f|) \text{ by the Triangle Inequality} \\
&\quad \text{and Monotonicity (Theorem 4.10)} \\
&= \int_{E\setminus A} |f_n - f| + \int_A |f_n| + \int_A |f| \text{ by linearity.} \qquad (29)
\end{aligned}$$

The Vitali Convergence Theorem (continued 2)

Proof (continued). Let $\varepsilon > 0$. Since $\{f_n\}$ is uniformly integrable, there is $\delta > 0$ such that

$$\int_A |f_n| < \varepsilon/3 \text{ for all } n \in \mathbb{N}$$

$$\text{and any measurable } A \subseteq E \text{ with } m(A) < \delta. \qquad (*)$$

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Since f is real-valued and E has finite measure, then by Egoroff's Theorem, there is $E_0 \subseteq E$ with $m(E_0) < \delta$ and $\{f_n\} \to f$ uniformly on $E \setminus E_0$. So there is $N \in \mathbb{N}$ such that $|f_n - f| < \varepsilon/(3m(E))$ on $E \setminus E_0$ for all $n \geq N$.

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Proof (continued). With $A = E_0$ in (29) we have for $n \geq N$,

$$\begin{aligned} \left| \int_E f_n - \int_E f \right| &\leq \int_{E \setminus E_0} |f_n - f| + \int_{E_0} |f_n| + \int_{E_0} |f| \\ &< \left(\frac{\varepsilon}{3m(E)} \right) m(E \setminus E_0) + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \text{ by } (*) \text{ and } (**) \\ &\leq \varepsilon. \end{aligned}$$

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Theorem 4.26. Let E be of finite measure. Suppose $\{h_n\}$ is a sequence of nonnegative integrable functions that converges pointwise a.e. on E to $h \equiv 0$. Then

$$\lim_{n\to\infty}\left(\int_E h_n\right) = 0 \text{ if and only if } \{h_n\} \text{ is uniformly integrable over } E.$$

Proof. If $\{h_n\}$ is uniformly integrable, then by the Vitali Convergence Theorem, $\lim_{n\to\infty}(\int_E h_n) = \int_E (\lim_{n\to\infty} h_n) = \int_E 0 = 0$.

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$$\int_A h_n \leq \int_E h_n < \varepsilon \text{ for } n \geq N. \qquad (30)$$

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$$\lim_{n\to\infty}\left(\int_E h_n\right) = 0 \text{ if and only if } \{h_n\} \text{ is uniformly integrable over } E.$$

Proof (continued). By Proposition 4.24, since each h_n is integrable, $\{h_n\}_{n=1}^{N-1}$ is uniformly integrable over E. Let $\delta > 0$ correspond to $\varepsilon > 0$ for this set in the definition of uniformly integrable. Then, trivially by (30), this $\delta > 0$ also works for all h_n with $n \geq N$. Therefore $\{h_n\}_{n=1}^{\infty}$ is uniformly integrable. □

Theorem 4.26 (continued)

Theorem 4.26. Let E be of finite measure. Suppose $\{h_n\}$ is a sequence of nonnegative integrable functions that converges pointwise a.e. on E to $h \equiv 0$. Then

$$\lim_{n\to\infty}\left(\int_E h_n\right) = 0 \text{ if and only if } \{h_n\} \text{ is uniformly integrable over } E.$$

Proof (continued). By Proposition 4.24, since each h_n is integrable, $\{h_n\}_{n=1}^{N-1}$ is uniformly integrable over E. Let $\delta > 0$ correspond to $\varepsilon > 0$ for this set in the definition of uniformly integrable. Then, trivially by (30), this $\delta > 0$ also works for all h_n with $n \geq N$. Therefore $\{h_n\}_{n=1}^{\infty}$ is uniformly integrable. □