Chapter 4. Lebesgue Integration
4.6. Uniform Integrability: The Vitali Convergence Theorem—Proofs of Theorems
Lemma 4.22. Let $E$ be a set of finite measure and $\delta > 0$. Then $E$ is the disjoint union of a finite collection of sets, each of which has measure less than $\delta$.

Proof. By the Continuity of Measure (Theorem 2.15(ii),

$$\lim_{n \to \infty} m(E \setminus [-n, n]) = \lim_{n \to \infty} (E \cap ((-\infty, -n) \cup (n, \infty))) = m(\emptyset) = 0,$$

since $E \cap ((-\infty, -n) \cup (n, \infty))$ is a descending collection of sets).
Lemma 4.22. Let $E$ be a set of finite measure and $\delta > 0$. Then $E$ is the disjoint union of a finite collection of sets, each of which has measure less than $\delta$.

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since $E \cap ((-\infty, -n) \cup (n, \infty))$ is a descending collection of sets). So by the definition of limit, there exists $n_0 \in \mathbb{N}$ for which $m(E \setminus [-n_0, n_0]) < \delta$. Now partition $[-n_0, n_0]$ into subintervals of length less than $\delta$, say producing intervals $I_1, I_2, \ldots, I_N$. 


Lemma 4.22. Let $E$ be a set of finite measure and $\delta > 0$. Then $E$ is the disjoint union of a finite collection of sets, each of which has measure less than $\delta$.

Proof. By the Continuity of Measure (Theorem 2.15(ii),

$$\lim_{n \to \infty} m(E \setminus [-n, n]) = \lim_{n \to \infty} (E \cap ((-\infty, -n) \cup (n, \infty))) = m(\emptyset) = 0,$$

since $E \cap ((-\infty, -n) \cup (n, \infty))$ is a descending collection of sets). So by the definition of limit, there exists $n_0 \in \mathbb{N}$ for which $m(E \setminus [-n_0, n_0]) < \delta$. Now partition $[-n_0, n_0]$ into subintervals of length less than $\delta$, say producing intervals $I_1, I_2, \ldots, I_N$. Then $E$ is the disjoint union of $E \setminus [-n_0, n_0] = E \cap ((-\infty, -n) \cap (n, \infty)), E \cap I_1, E \cap I_2, \ldots, E \cap I_N$, and each is of measure less than $\delta$. \qed
Lemma 4.22. Let $E$ be a set of finite measure and $\delta > 0$. Then $E$ is the disjoint union of a finite collection of sets, each of which has measure less than $\delta$.

Proof. By the Continuity of Measure (Theorem 2.15(ii),

$$\lim_{n \to \infty} m(E \setminus [-n, n]) = \lim_{n \to \infty} (E \cap ((-\infty, -n) \cup (n, \infty))) = m(\emptyset) = 0,$$

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Proposition 4.23. Let $f$ be a measurable function on $E$. If $f$ is integrable over $E$, then for each $\varepsilon > 0$, there is $\delta > 0$ for which:

$$\text{if } A \subseteq E \text{ is measurable and } m(A) < \delta \text{ then } \int_A |f| < \varepsilon.$$  \hfill (26)

Conversely, for $m(E) < \infty$, if for each $\varepsilon > 0$, there is a $\delta > 0$ for which (26) holds, then $f$ is integrable over $E$.

Proof. WLOG, we assume $f \geq 0$ on $E$ (otherwise, we use an $\varepsilon/2$ argument on the positive and negative parts of $f$). Assume $f$ is integrable over $E$ and let $\varepsilon > 0$. 
Proposition 4.23. Let \( f \) be a measurable function on \( E \). If \( f \) is integrable over \( E \), then for each \( \varepsilon > 0 \), there is \( \delta > 0 \) for which:

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**Proof.** WLOG, we assume \( f \geq 0 \) on \( E \) (otherwise, we use an \( \varepsilon/2 \) argument on the positive and negative parts of \( f \)). Assume \( f \) is integrable over \( E \) and let \( \varepsilon > 0 \). Then, by the definition of the integral of a nonnegative integrable function of finite support (that is, nonzero on a set of finite measure), say \( f_\varepsilon \), for which \( 0 \leq f_\varepsilon \leq f \) on \( E \) and \( 0 \leq \int_E f - \int_E f_\varepsilon < \varepsilon/2 \) (recall that \( \int_E f \) is defined as a supremum over such \( f_\varepsilon \)).
Proposition 4.23.

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Proposition 4.23 (continued 1)

**Proof (continued).** If \( A \subseteq E \) is measurable then

\[
\int_A f - \int_A f_\varepsilon = \int_A (f - f_\varepsilon) \text{ by linearity}
\]

\[
\leq \int_E (f - f_\varepsilon) \text{ by additivity over } A \cup (E \setminus A) \text{ since } f - f_\varepsilon \geq 0
\]

\[
= \int_E f - \int_E f_\varepsilon \text{ by linearity}
\]

\[
< \frac{\varepsilon}{2} \text{ by above.}
\]

Since \( f_\varepsilon \) is bounded, choose \( M > 0 \) for which \( 0 \leq f_\varepsilon < M \) on \( E \). So for \( A \subseteq E \) measurable,

\[
\int_A f < \left( \int_A f_\varepsilon \right) + \frac{\varepsilon}{2} \text{ by the above inequality}
\]

\[
\leq Mm(A) + \frac{\varepsilon}{2}.
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Proof (continued). If $A \subseteq E$ is measurable then

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Proposition 4.23 (continued 2)

Proof (continued). Define $\delta = \varepsilon/(2M)$, and this inequality implies for $m(A) < \delta = \varepsilon/(2M)$ that $\int_A f < M(\varepsilon/(2M)) + \varepsilon/2 = \varepsilon$.

Conversely, suppose $m(E) < \infty$ and suppose that for all $\varepsilon > 0$, there is a $\delta > 0$ for which (26) holds. For $\varepsilon = 1$, let $\delta_0 > 0$ be the corresponding $\delta$. Since $m(E) < \infty$, by Lemma 4.22, we may express $E$ as the disjoint union of a finite collection of measurable subsets $\{E_k\}_{k=1}^N$, each of measure less than $\delta_0$. 

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**Proof (continued).** Define $\delta = \varepsilon/(2M)$, and this inequality implies for $m(A) < \delta = \varepsilon/(2M)$ that $\int_A f < M(\varepsilon/(2M)) + \varepsilon/2 = \varepsilon$.

Conversely, suppose $m(E) < \infty$ and suppose that for all $\varepsilon > 0$, there is a $\delta > 0$ for which (26) holds. For $\varepsilon = 1$, let $\delta_0 > 0$ be the corresponding $\delta$.

Since $m(E) < \infty$, by Lemma 4.22, we may express $E$ as the disjoint union of a finite collection of measurable subsets $\{E_k\}_{k=1}^N$, each of measure less than $\delta_0$. Therefore (by additivity and since $\varepsilon = 1$),

$$\int_E f = \int_{\bigcup E_k} f = \sum_{k=1}^N \left( \int_{E_k} f \right) < \sum_{k=1}^N (1) = N.$$ 

So $f$ is integrable over $E$. \qed
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**Proof (continued).** Define $\delta = \frac{\varepsilon}{2M}$, and this inequality implies for $m(A) < \delta = \frac{\varepsilon}{2M}$ that $\int_A f < M(\varepsilon/(2M)) + \varepsilon/2 = \varepsilon$.

Conversely, suppose $m(E) < \infty$ and suppose that for all $\varepsilon > 0$, there is a $\delta > 0$ for which (26) holds. For $\varepsilon = 1$, let $\delta_0 > 0$ be the corresponding $\delta$. Since $m(E) < \infty$, by Lemma 4.22, we may express $E$ as the disjoint union of a finite collection of measurable subsets $\{E_k\}_{k=1}^N$, each of measure less than $\delta_0$. Therefore (by additivity and since $\varepsilon = 1$),

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Proposition 4.25

**Proposition 4.25.** Assume $E$ has finite measure. Let the sequence of functions $\{f_k\}_{k=1}^{\infty}$ be uniformly integrable over $E$. If $\{f_n\} \to f$ pointwise a.e. on $E$, then $f$ is integrable over $E$.

**Proof.** First, by Proposition 3.9, $f$ is measurable.
Proposition 4.25. Assume $E$ has finite measure. Let the sequence of functions $\{f_k\}_{k=1}^{\infty}$ be uniformly integrable over $E$. If $\{f_n\} \to f$ pointwise a.e. on $E$, then $f$ is integrable over $E$.

Proof. First, by Proposition 3.9, $f$ is measurable. Let $\varepsilon = 1$ and let $\delta_0 > 0$ be the corresponding $\delta > 0$ given by the definition of uniform integrability. Since $E$ has finite measure, by Lemma 4.22, we may express $E$ as a disjoint union of a finite collection of measurable sets $\{E_k\}_{k=1}^{N}$ such that $m(E_k) < \delta_0$ for $1 \leq k \leq N$. \\

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\int_E |f_n| = \int_{\bigcup E_k} = \sum_{k=1}^{N} \left( \int_{E_k} |f_n| \right) < \sum_{k=1}^{N} (1) = N.
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\int_E |f_n| = \int_{\bigcup E_k} = \sum_{k=1}^{N} \left( \int_{E_k} |f_n| \right) < \sum_{k=1}^{N} (1) = N.
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So, by Fatou’s Lemma, $\int_E |f| \leq \lim \inf \int_E |f_n| \leq N$. So $|f|$ is integrable over $E$ and so (by definition, see page 85) $f$ is integrable over $E$. \qed
Proposition 4.25. Assume $E$ has finite measure. Let the sequence of functions $\{f_k\}_{k=1}^{\infty}$ be uniformly integrable over $E$. If $\{f_n\} \rightarrow f$ pointwise a.e. on $E$, then $f$ is integrable over $E$.

**Proof.** First, by Proposition 3.9, $f$ is measurable. Let $\varepsilon = 1$ and let $\delta_0 > 0$ be the corresponding $\delta > 0$ given by the definition of uniform integrability. Since $E$ has finite measure, by Lemma 4.22, we may express $E$ as a disjoint union of a finite collection of measurable sets $\{E_k\}_{k=1}^{N}$ such that $m(E_k) < \delta_0$ for $1 \leq k \leq N$. Then (by additivity and since $\varepsilon = 1$)

$$\int_{E} |f_n| = \int_{\bigcup E_k} = \sum_{k=1}^{N} \left( \int_{E_k} |f_n| \right) < \sum_{k=1}^{N} (1) = N.$$

So, by Fatou’s Lemma, $\int_{E} |f| \leq \lim \inf \int_{E} |f_n| \leq N$. So $|f|$ is integrable over $E$ and so (by definition, see page 85) $f$ is integrable over $E$. \qed
The Vitali Convergence Theorem

Let $E$ be of finite measure. Suppose sequence $\{f_n\}$ is uniformly integrable over $E$. If $\{f_n\} \to f$ pointwise a.e. on $E$, then $f$ is integrable over $E$ and

$$\lim_{n \to \infty} \left( \int_E f_n \right) = \int_E \left( \lim_{n \to \infty} f_n \right) = \int_E f.$$ 

**Proof.** By Proposition 4.15, $f$ is integrable over $E$. So by Proposition 4.15, $f$ is finite a.e. on $E$. 
The Vitali Convergence Theorem

Let $E$ be of finite measure. Suppose sequence $\{f_n\}$ is uniformly integrable over $E$. If $\{f_n\} \to f$ pointwise a.e. on $E$, then $f$ is integrable over $E$ and

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Proof. By Proposition 4.15, $f$ is integrable over $E$. So by Proposition 4.15, $f$ is finite a.e. on $E$. Also by Proposition 4.15, we can “excise” a subset of $E$ of measure 0 (on which the pointwise convergence does not hold) and assume WLOG that $f$ is real valued (as opposed to extended real valued) and that the convergence is pointwise on all of $E$. 
The Vitali Convergence Theorem.

Let $E$ be of finite measure. Suppose sequence $\{f_n\}$ is uniformly integrable over $E$. If $\{f_n\} \to f$ pointwise a.e. on $E$, then $f$ is integrable over $E$ and

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The Vitali Convergence Theorem (continued 1)

Proof (continued). For measurable $A \subseteq E$ and $n \in \mathbb{N}$

$$\left| \int_E f_n - \int_E f \right| = \left| \int_E (f_n - f) \right| \text{ by linearity}$$

$$\leq \int_E |f_n - f| \text{ by Proposition 4.16, the Integral Comparison Test}$$

$$= \int_{E\setminus A} |f_n - f| + \int_A |f_n - f| \text{ by additivity}$$

$$\leq \int_{E\setminus A} |f_n - f| + \int_A (|f_n| + |f|) \text{ by the Triangle Inequality and Monotonicity}$$

$$= \int_{E\setminus A} |f_n - f| + \int_A |f_n| + \int_A |f| \text{ by linearity.} \quad (29)$$
Proof (continued). Let $\varepsilon > 0$. Since $\{f_n\}$ is uniformly integrable, there is $\delta > 0$ such that

$$\int_A |f_n| < \varepsilon / 3 \quad \text{for all } n \in \mathbb{N}$$

and any measurable $A \subseteq E$ with $m(A) < \delta$. \hfill (*)
Proof (continued). Let $\varepsilon > 0$. Since $\{f_n\}$ is uniformly integrable, there is $\delta > 0$ such that

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By Fatou’s Lemma

$$\int_A |f| \leq \liminf \int_A |f_n| \leq \frac{\varepsilon}{3}. \quad (** \hfill$$
The Vitali Convergence Theorem (continued 2)

Proof (continued). Let $\varepsilon > 0$. Since $\{f_n\}$ is uniformly integrable, there is $\delta > 0$ such that

$$\int_A |f_n| < \varepsilon/3 \text{ for all } n \in \mathbb{N}$$

and any measurable $A \subseteq E$ with $m(A) < \delta$. ($\ast$)

By Fatou’s Lemma

$$\int_A |f| \leq \lim \inf \int_A |f_n| \leq \frac{\varepsilon}{3}. \text{ (**)}$$

Since $f$ is real-valued and $E$ has finite measure, then by Egoroff’s Theorem, there is $E_0 \subseteq E$ with $m(E_0) < \delta$ and $\{f_n\} \to f$ uniformly on $E \setminus E_0$. So there is $N \in \mathbb{N}$ such that $|f_n - f| < \varepsilon/(3m(E))$ on $E \setminus E_0$ for all $n \geq N$. 
Proof (continued). Let $\varepsilon > 0$. Since $\{f_n\}$ is uniformly integrable, there is $\delta > 0$ such that

$$\int_A |f_n| < \frac{\varepsilon}{3} \text{ for all } n \in \mathbb{N}$$

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$$\int_A |f| \leq \liminf \int_A |f_n| \leq \frac{\varepsilon}{3}. \quad (***)$$

Since $f$ is real-valued and $E$ has finite measure, then by Egoroff’s Theorem, there is $E_0 \subseteq E$ with $m(E_0) < \delta$ and $\{f_n\} \to f$ uniformly on $E \setminus E_0$. So there is $N \in \mathbb{N}$ such that $|f_n - f| < \frac{\varepsilon}{3m(E)}$ on $E \setminus E_0$ for all $n \geq N$. 

Proof (continued). With $A = E_0$ in (29) we have for $n \geq N$,

$$\left| \int_E f_n - \int_E f \right| \leq \int_{E \setminus E_0} |f_n - f| + \int_{E_0} |f_n| + \int_{E_0} |f|$$

$$< \left( \frac{\varepsilon}{3m(E)} \right) m(E \setminus E_0) + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \text{ by (*) and (**) }$$

$$\leq \varepsilon.$$

So $\int_E f_n \to \int_E f$. \qed
Proof (continued). With $A = E_0$ in (29) we have for $n \geq N,$

$$\left| \int_E f_n - \int_E f \right| \leq \int_{E \setminus E_0} |f_n - f| + \int_{E_0} |f_n| + \int_{E_0} |f|$$

$$< \left( \frac{\varepsilon}{3m(E)} \right) m(E \setminus E_0) + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \text{ by (*) and (**) }$$

$$\leq \varepsilon.$$ 

So $\int_E f_n \to \int_E f.$
Theorem 4.26

**Theorem 4.26.** Let $E$ be of finite measure. Suppose $\{h_n\}$ is a sequence of nonnegative integrable functions that converges pointwise a.e. on $E$ to $h \equiv 0$. Then

$$\lim_{n \to \infty} \left( \int_E h_n \right) = 0 \text{ if and only if } \{h_n\} \text{ is uniformly integrable over } E.$$

**Proof.** If $\{h_n\}$ is uniformly integrable, then by the Vitali Convergence Theorem, $\lim_{n \to \infty} (\int_E h_n) = 0$. 
Theorem 4.26

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**Proof.** If $\{h_n\}$ is uniformly integrable, then by the Vitali Convergence Theorem, $\lim_{n \to \infty} (\int_E h_n) = 0$.

Conversely, suppose $\lim_{n \to \infty} (\int_E h_n) = 0$ and let $\varepsilon > 0$. Then there is $N \in \mathbb{N}$ for which $\int_E h_n < \varepsilon$ for $n \geq N$. 
Theorem 4.26. Let $E$ be of finite measure. Suppose $\{h_n\}$ is a sequence of nonnegative integrable functions that converges pointwise a.e. on $E$ to $h \equiv 0$. Then

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Conversely, suppose $\lim_{n \to \infty} (\int_E h_n) = 0$ and let $\varepsilon > 0$. Then there is $N \in \mathbb{N}$ for which $\int_E h_n < \varepsilon$ for $n \geq N$. Since $h_n \geq 0$ on $E$, then if $A \subseteq E$ is measurable and $n \geq N$ then by monotonicity,

$$\int_A h_n \leq \int_E h_n < \varepsilon \text{ for } n \geq N. \quad (30)$$
Theorem 4.26. Let $E$ be of finite measure. Suppose $\{h_n\}$ is a sequence of nonnegative integrable functions that converges pointwise a.e. on $E$ to $h \equiv 0$. Then

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Theorem 4.26. Let $E$ be of finite measure. Suppose $\{h_n\}$ is a sequence of nonnegative integrable functions that converges pointwise a.e. on $E$ to $h \equiv 0$. Then

$$\lim_{n \to \infty} \left( \int_E h_n \right) = 0$$
if and only if $\{h_n\}$ is uniformly integrable over $E$.

Proof (continued). By Proposition 4.24, since each $h_n$ is integrable, $\{h_n\}_{n=1}^{N-1}$ is uniformly integrable over $E$. Let $\delta > 0$ correspond to $\varepsilon > 0$ for this set in the definition of uniformly integrable. Then, trivially by (30), this $\delta > 0$ also works for all $h_n$ with $n \geq N$. Therefore $\{h_n\}_{n=1}^{\infty}$ is uniformly integrable.
Theorem 4.26. Let $E$ be of finite measure. Suppose $\{h_n\}$ is a sequence of nonnegative integrable functions that converges pointwise a.e. on $E$ to $h \equiv 0$. Then

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