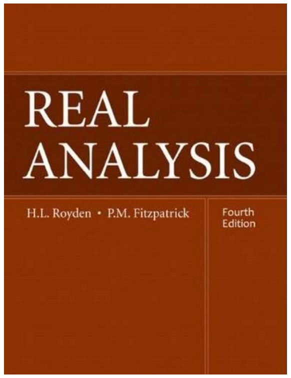
Real Analysis

Chapter 5. Lebesgue Integration: Further Topics

5.1. Uniform Integrability and Tightness: A General Vitali Convergence Theorem—Proofs of Theorems



Proposition 5.1

Proposition 5.1. Let f be integrable over E. Then for each $\varepsilon > 0$, there is a set of finite measure E_0 for which $\int_{E \setminus E_0} |f| < \varepsilon$.

Proof. Let $\varepsilon > 0$. The fact that f is integrable over E implies (by definition) that $|f|$ is integrable over E. By the definition of $\int_E |f|$, there is a bounded measurable function g on E, which vanishes outside a set $E_0 \subseteq E$ of finite measure for which $0 \le g \le |f|$ and $\int_E |f| - \int_E g < \varepsilon$ (since $\int_E |f|$ is defined as in terms of a supremum involving integrals of such functions). So

$$\begin{aligned} \int_{E\setminus E_0} |f| &= \int_{E\setminus E_0} (|f| - g) \text{ since } g \text{ vanishes outside } E_0 \\ &\le \int_{E\setminus E_0} (|f| - g) + \int_{E_0} (|f| - g) \text{ since } |f| - g \text{ is nonnegative} \\ &= \int_E (|f| - g) < \varepsilon \text{ by additivity, Theorem 4.11, as claimed.} \end{aligned}$$

□

The General Vitali Convergence Theorem

The General Vitali Convergence Theorem.
Let $\{f_n\}$ be a sequence of functions on E that is uniformly integrable and tight over E. Suppose $\{f_n\} \to f$ pointwise a.e. on E. Then f is integrable over E and

$$\lim_{n\to\infty} \left(\int_E f_n \right) = \int_E \left(\lim_{n\to\infty} f_n \right) = \int_E f.$$

Proof. Let $\varepsilon > 0$. Since $\{f_n\}$ is tight over E, there is measurable $E_0 \subseteq E$ of finite measure for which $\int_{E\setminus E_0} |f_n| < \varepsilon/4$ for all $n \in \mathbb{N}$. By Fatou's Lemma, $\int_{E\setminus E_0} |f| \le \liminf \left(\int_{E\setminus E_0} |f_n| \right) \le \varepsilon/4$. So f is integrable over $E \setminus E_0$. Also

$$\left| \int_{E\setminus E_0} (f_n - f) \right| \le \int_{E\setminus E_0} |f_n - f| \text{ by Proposition 4.16,}$$

Integral Comparison Test...

The General Vitali Convergence Theorem (continued 1)

Proof (continued). ...

$$\begin{aligned} \left| \int_{E\setminus E_0} (f_n - f) \right| &\le \int_{E\setminus E_0} |f_n - f| \\ &\le \int_{E\setminus E_0} |f_n| + \int_{E\setminus E_0} |f| \text{ by the Triangle Inequality} \\ &\quad \text{and monotonicity (Theorem 4.10)} \\ &< \varepsilon/4 + \varepsilon/4 = \varepsilon/2. \end{aligned}$$

Since $m(E_0) < \infty$ and $\{f_n\}$ is uniformly integrable over E_0 then each f_n is integrable over E_0 by Proposition 4.23, and by the Vitali Convergence Theorem f is integrable over E_0 and $\lim_{n\to\infty} \left(\int_{E_0} f_n \right) = \int_{E_0} f$. So there is $N \in \mathbb{N}$ such that $|\int_{E_0} f_n - \int_{E_0} f| = |\int_{E_0} (f_n - f)| < \varepsilon/2$ for all $n \ge N$ (by linearity, Theorem 4.17). Since f is integrable over $E \setminus E_0$ and E_0, then f is integrable over E. Each f_n is integrable over E by Note 5.1.A.

The General Vitali Convergence Theorem (continued 2)

Proof (continued). Combining the above,

$$\begin{aligned}
\left| \int_E f_n - \int_E f \right| &= \left| \int_E (f_n - f) \right| \text{ by linearity, Theorem 4.17} \\
&= \left| \int_{E \setminus E_0} (f_n - f) + \int_{E_0} (f_n - f) \right| \text{ by additivity, Cor. 4.18} \\
&\leq \left| \int_{E \setminus E_0} (f_n - f) \right| + \left| \int_{E_0} (f_n - f) \right| \text{ by Triangle Inequality} \\
&< \varepsilon/2 + \varepsilon/2 \text{ for } n \geq N \\
&= \varepsilon.
\end{aligned}$$

So $\lim_{n\to\infty}(\int_E f_n) = \int_E f$. □