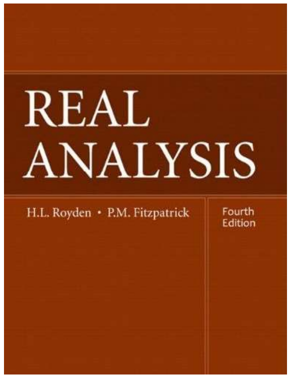
Real Analysis

Chapter 5. Lebesgue Integration: Further Topics

5.2. Convergence in Measure—Proofs of Theorems



Proposition 5.3

Proposition 5.3. Assume E has finite measure. Let $\{f_n\}$ be a sequence of measurable functions on E that converges pointwise a.e. on E to f, and f is finite a.e. on E. Then $\{f_n\} \to f$ in measure on E.

Proof. First, f is measurable since it is the a.e. pointwise limit of a sequence of measurable functions (Proposition 3.9). Let $\eta > 0$. Let $\varepsilon > 0$. By Egoroff's Theorem, there is measurable $F \subseteq E$ with $m(E \setminus F) < \varepsilon$ such that $\{f_n\} \to f$ uniformly on F. So there exists $N \in \mathbb{N}$ such that for all $n \geq N$ we have $|f_n - f| < \eta$ on F. So for $n \geq N$, $\{x \in E \mid |f_n(x) - f(x)| > \eta\} \subseteq E \setminus F$ and $m(\{x \in E \mid |f_n(x) - f(x)| > \eta\}) < \varepsilon$ since $m(E \setminus F) < \varepsilon$. So $\lim_{n\to\infty} m(\{x \in E \mid |f_n(x) - f(x)| > \eta\}) = 0$ and $\{f_n\} \to f$ in measure, by definition. □

Theorem 5.4. Riesz

Theorem 5.4. (Riesz.)
If $\{f_n\} \to f$ in measure on E, then there is a subsequence $\{f_{n_k}\}$ that converges pointwise a.e. on E to f.

Proof. Let $k \in \mathbb{N}$. Since $\{f_n\} \to f$ in measure, for $\eta = 1/k$ we have $\lim_{n\to\infty} m(\{x \in E \mid |f_n(x) - f(x)| > 1/k\}) = 0$ and for $\varepsilon = 1/2^k$, there exists $n_k \in \mathbb{N}$ such that for all $j \geq n_k$, $m(\{x \in E \mid |f_j(x) - f(x)| > 1/k\}) < 1/2^k$. Define $E_k = \{x \in E \mid |f_{n_k}(x) - f(x)| > 1/k\}$. Then $m(E_k) < 1/2^k$ and so $\sum_{k=1}^\infty m(E_k) < 1 < \infty$. The Borel-Cantelli Lemma implies that almost all $x \in E$ lie in finitely many E_k. That is, for almost all $x \in E$, there is index $K(x)$ such that $x \notin E_k$ if $k \geq K(x)$, or that $|f_{n_k}(x) - f(x)| \leq 1/k$ for all $k \geq K(x)$. So for such x (i.e., almost all $x \in E$) we have $\lim_{k\to\infty} f_{n_k}(x) = f(x)$; that is, the subsequence $\{f_{n_k}\} \to f$ a.e. on E. □

Corollary 5.5

Corollary 5.5. Let $\{f_n\}$ be a sequence of nonnegative integrable functions on E. Then $\displaystyle\lim_{n\to\infty}\int_E f_n = 0$ if and only if: $\{f_n\} \to 0$ in measure on E and $\{f_n\}$ is uniformly integrable and tight over E.

Proof. Suppose $\displaystyle\lim_{n\to\infty}\int_E f_n = 0$. Then by Corollary 5.2 (which requires that $\{f_n(x)\} \to 0$ for almost all $x \in E$, but this is required to show the part of the result which implies $\lim_{n\to\infty} \int_E f_n = 0$ and we are taking this as our hypothesis here), we have that $\{f_n\}$ is uniformly integrable and tight over E, as claimed. To show that $\{f_n\} \to 0$ in measure on E, let $\eta > 0$. By Chebychev's Inequality, for each $n \in \mathbb{N}$ we have

$$m(\{x \in E \mid f_n > \eta\}) \leq \frac{1}{\eta}\int_E f_n.$$

So $0 \leq \lim_{n\to\infty} m(\{x \in E \mid f_n > \eta\}) \leq \frac{1}{\eta}\lim_{n\to\infty}\int_E f_n = 0$ and, by definition, $\{f_n\} \to 0$ in measure on E.

Corollary 5.5 (continued)

Proof (continued). To prove the converse, suppose $\{f_n\} \to 0$ in measure on E and $\{f_n\}$ is uniformly integrable and tight over E. ASSUME $\lim_{n\to\infty} \int_E f_n \neq 0$. Then there is some $\varepsilon_0 > 0$ and a subsequence $\{f_{n_k}\}$ for which

$$\int_E f_{n_k} \geq \varepsilon_0 \text{ for all } k \in \mathbb{N}. \qquad (*)$$

However, by Theorem 5.4 some subsequence $\{f_{n_{k_j}}\}$ of $\{f_{n_k}\}$ converges to $f \equiv 0$ pointwise a.e. on E and this subsequence of the original sequence $\{f_n\}$ is uniformly integrable and tight. But then by the Generalized Vitali Convergence Theorem $\lim_{j\to\infty} \int_E f_{n_{k_j}} = \int_E f = 0$. This is a CONTRADICTION to $(*)$, so that the assumption $\lim_{n\to\infty} \int_E f_n \neq 0$ is false, and hence $\lim_{n\to\infty} \int_E f_n = 0$, as claimed. □