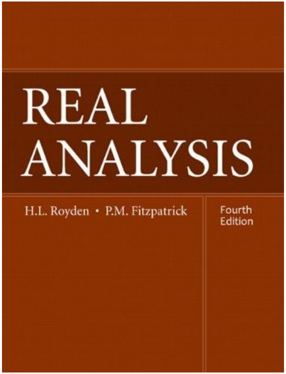
Real Analysis

Chapter 5. Lebesgue Integration: Further Topics

5.2. Convergence in Measure—Proofs of Theorems



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Proof. First, f is measurable since it is the a.e. pointwise limit of a sequence of measurable functions (Proposition 3.9). Let $\eta > 0$. Let $\varepsilon > 0$. By Egoroff's Theorem, there is measurable $F \subseteq E$ with $m(E \setminus F) < \varepsilon$ such that $\{f_n\} \to f$ uniformly on F.

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If $\{f_n\} \to f$ in measure on E, then there is a subsequence $\{f_{n_k}\}$ that converges pointwise a.e. on E to f.

Proof. Let $k \in \mathbb{N}$. Since $\{f_n\} \to f$ in measure, for $\eta = 1/k$ we have $\lim_{n\to\infty} m(\{x \in E \mid |f_n(x) - f(x)| > 1/k\}) = 0$ and for $\varepsilon = 1/2^k$, there exists $n_k \in \mathbb{N}$ such that for all $j \geq n_k$, $m(\{x \in E \mid |f_j(x) - f(x)| > 1/k\}) < 1/2^k$.

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Corollary 5.5. Let $\{f_n\}$ be a sequence of nonnegative integrable functions on E. Then $\lim_{n\to\infty}\int_E f_n = 0$ if and only if: $\{f_n\} \to 0$ in measure on E and $\{f_n\}$ is uniformly integrable and tight over E.

Proof. Suppose $\lim_{n\to\infty}\int_E f_n = 0$. Then by Corollary 5.2 (which requires that $\{f_n(x)\} \to 0$ for almost all $x \in E$, but this is required to show the part of the result which implies $\lim_{n\to\infty}\int_E f_n = 0$ and we are taking this as our hypothesis here), we have that $\{f_n\}$ is uniformly integrable and tight over E, as claimed.

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$$m(\{x \in E \mid f_n > \eta\}) \leq \frac{1}{\eta} \int_E f_n.$$

So $0 \leq \lim_{n\to\infty} m(\{x \in E \mid f_n > \eta\}) \leq \frac{1}{\eta} \lim_{n\to\infty} \int_E f_n = 0$ and, by definition, $\{f_n\} \to 0$ in measure on E.

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$$\int_E f_{n_k} \geq \varepsilon_0 \text{ for all } k \in \mathbb{N}. \qquad (*)$$

However, by Theorem 5.4 some subsequence $\{f_{n_{k_j}}\}$ of $\{f_{n_k}\}$ converges to $f \equiv 0$ pointwise a.e. on E and this subsequence of the original sequence $\{f_n\}$ is uniformly integrable and tight. But then by the Generalized Vitali Convergence Theorem $\lim_{j\to\infty} \int_E f_{n_{k_j}} = \int_E f = 0$.

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