Lemma 5.6

Lemma 5.6. Let \( \{\varphi_n\} \) and \( \{\psi_n\} \) be sequences of functions, each of which is integrable over \( E \), such that \( \{\varphi_n\} \) is increasing while \( \{\psi_n\} \) is decreasing on \( E \). Let the function \( i \) on \( E \) have the property \( \varphi_n \leq f \leq \psi_n \) on \( E \) for all \( n \). If \( \lim_{n \to \infty} \left( \int_E (\psi_n - \varphi_n) \right) = 0 \), then \( \{\varphi_n\} \to f \) pointwise a.e. on \( E \), \( \{\psi_n\} \to f \) pointwise a.e. on \( E \), \( f \) is integrable over \( E \).

\[
\lim_{n \to \infty} \left( \int_E \varphi_n \right) = \int_E f \quad \text{and} \quad \lim_{n \to \infty} \left( \int_E \psi_n \right) = \int_E f.
\]

Proof. For \( x \in E \), define \( \varphi^*(x) = \lim_{n \to \infty} \varphi_n(x) \) and \( \psi^*(x) = \lim_{n \to \infty} \psi_n(x) \). The functions \( \varphi^* \) and \( \psi^* \) are well defined since \( \{\varphi_n\} \) and \( \{\psi_n\} \) are monotone and monotone sequence of extended real numbers converge (to an extended real number). \( \varphi^* \) and \( \psi^* \) are measurable by Proposition 3.9.

Lemma 5.6 (continued 1)

Proof (continued). By monotonicity and properties of limits, \( \varphi_n \leq \varphi^* \leq f \leq \psi^* \leq \psi_n \) on \( E \) and for all \( n \). So \( 0 \leq \psi^* - \varphi^* \leq \psi_n - \varphi_n \) for all \( n \) and by monotonicity of integrals \( 0 \leq \int_E (\psi^* - \varphi^*) \leq \int_E (\psi_n - \varphi_n) \) for all \( n \), so \( 0 \leq \int_E (\psi^* - \varphi^*) \leq \lim_{n \to \infty} (\int_E (\psi_n - \varphi_n)) = 0 \). Since \( \psi^* - \varphi^* \) is nonnegative and measurable and \( \int_E (\psi^* - \varphi^*) = 0 \), then Proposition 4.9 tells us that \( \psi^* = \varphi^* \) a.e. on \( E \). But \( \varphi^* \leq f \leq \psi^* \) on \( E \), so \( \{\varphi^*\} \to f \) and \( \{\psi_n\} \to f \) pointwise a.e. on \( E \). So \( f \) is measurable (because \( f = \varphi^* = \psi^* \) a.e. on \( E \) and \( \varphi^*, \psi^* \) are measurable). Next, \( 0 \leq f - \varphi_1 \leq \psi_1 - \varphi_1 \) on \( E \) (\( \{\psi_n\} \) is decreasing), and \( \psi_1 \) and \( \varphi_1 \) are integrable over \( E \), so by the Integral Comparison Test (Theorem 4.16), \( f \) is integrable over \( E \).

Lemma 5.6 (continued 2)

Proof (continued). By the above inequalities and monotonicity, \( 0 \leq \int_E \psi_n - \int_E f = \int_E (\psi_n - f) \leq \int (\psi_n - \varphi_n) \) (also by linearity and the fact that \( \{\varphi_n\} \) is increasing) and 
\[
0 \leq \int_E f - \int_E \varphi_n = \int_E (f - \varphi_n) \leq \int_E (\psi_n - \varphi_n) \quad (\{\varphi_n\} \text{ is decreasing})
\]
and therefore \( \lim_{n \to \infty} (\int_E \varphi_n) = \int_E f = \lim_{n \to \infty} (\int_E \psi_n) \) since (by the Lebesgue Dominated Convergence Theorem) \( \lim_{n \to \infty} \int_E (\psi_n - \varphi_n) = \int_E \lim_{n \to \infty} (\psi_n - \varphi_n) = 0 \). \( \square \)
Theorem 5.7

**Theorem 5.7.** Let $f$ be a bounded function on a set of finite measure $E$. Then $f$ is Lebesgue integrable over $E$ if and only if $f$ is measurable.

**Proof.** If $f$ is measurable, then $f$ is Lebesgue integrable by Theorem 4.4. Suppose $f$ is Lebesgue integrable over $E$ (that is, the upper and lower Lebesgue integrals are equal). Since

$$
\int_E f = \sup \left\{ \int_E \varphi \mid \varphi \text{ is simple, } \varphi \leq f \text{ on } E \right\},
$$

$$
\int_E f = \inf \left\{ \int_E \psi \mid \psi \text{ is simple, } \varphi \geq f \text{ on } E \right\},
$$

then by the definition of sup and inf, there are sequences of simple functions, $\{\varphi_n\}$ and $\{\psi_n\}$ such that $\varphi_n \leq f \leq \psi_n$ on $E$ for all $n$, $\int_E f = \lim (\int_E \varphi_n)$, and $\int_E f = \lim (\int_E \psi_n)$.

\[\text{(continued)}\]

**Theorem 5.7 (continued)**

**Theorem 5.7.** Let $f$ be a bounded function on a set of finite measure $E$. Then $f$ is Lebesgue integrable over $E$ if and only if $f$ is measurable.

**Proof (continued).** So

$$
0 = \int_E f - \int_E f = \lim \left( \int_E \varphi_n \right) - \lim \left( \int_E \varphi_n \right)
$$

$$
= \lim \left( \int_E \psi_n - \int_E \varphi_n \right)
$$

$$
= \lim \int_E (\psi_n - \varphi_n) \text{ by linearity.}
$$

Since the max and min of a pair of simple functions are again simple, using the monotonicity of integration and possibly replacing $\varphi_n$ by $\max_{1 \leq i \leq n} \{\varphi_i\}$ and $\psi_n$ by $\min_{1 \leq i \leq n} \{\psi_i\}$ (pointwise) we may suppose $\{\varphi_n\}$ is increasing and $\{\psi_n\}$ is decreasing. By Lemma 5.6, $\{\varphi_n\} \to f$ pointwise a.e. on $E$ (so does $\{\psi_n\}$). So $f$ is measurable since it is the pointwise limit a.e. of a sequence of measurable functions (by Proposition 3.9). \qed